# Interpolation via Barycentric Coordinates

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Some material from D. Anisimov

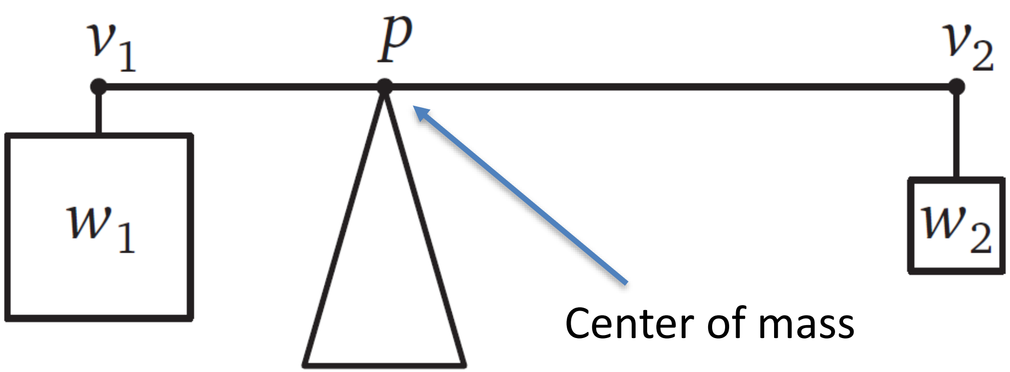
# Outline

- Barycenter
- Convexity
- Barycentric coordinates
  - For Simplices
  - For point sets
    - Inverse distance (Shepard)
    - Delaunay / Voronoi
    - Natural neighbors (Sibson)
  - For convex polyhedra
  - Generalized barycentric coordinates
  - Applications

# BARYCENTER

# Barycenter

- Law of lever: $w_1 l_1 = w_2 l_2$




$$l_1 = p - v_1$$
$$l_2 = v_2 - p$$

# Barycentric coordinates

$$w_1(v_1 - p) + w_2(v_2 - p) = 0$$

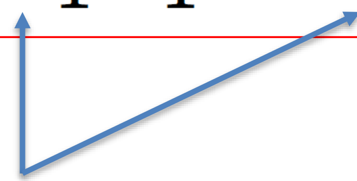
$$w_1 v_1 + w_2 v_2 = W p$$

$$w_1 + w_2$$

$$\boxed{b_1 v_1 + b_2 v_2 = p}$$

# Opposite problem?

$$b_1 v_1 + b_2 v_2 = p$$



?

# Opposite problem

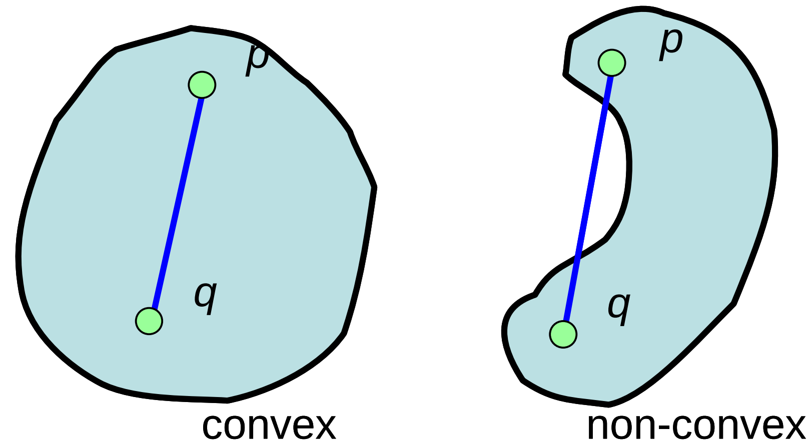
$$b_1 v_1 + b_2 v_2 = p$$

$l_2/l$ $\quad$ $l_1/l$ $\quad$ unique!

# CONVEXITY

# Convexity

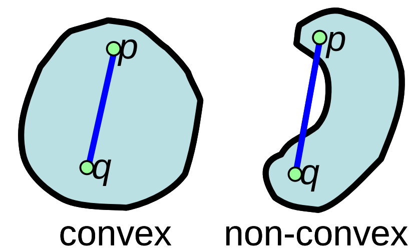
A set $S$ is *convex* if any pair of points $p,q \in S$ satisfy $pq \subseteq S$.

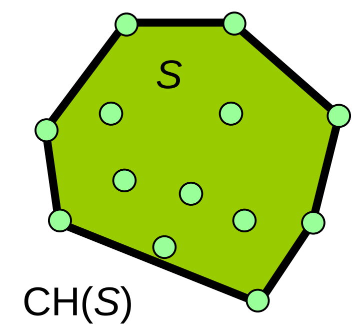


# Convex Hull

- The *convex hull* of a set *S* is:
  - The minimal convex set that contains *S*, i.e. any convex set *C* such that $S \subseteq C$ satisfies $CH(S) \subseteq C$.
  - The intersection of all convex sets that contain *S*.
  - The set of all convex combinations of $p_i \in S$, i.e. all points of the form:

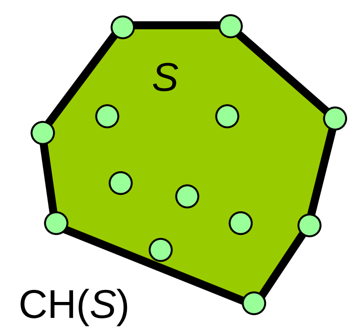
$$\sum_{i=1}^{n} \alpha_i p_i \ , \qquad \alpha_i \geq 0, \ \sum_{i=1}^{n} \alpha_i = 1$$



# Convex Hulls

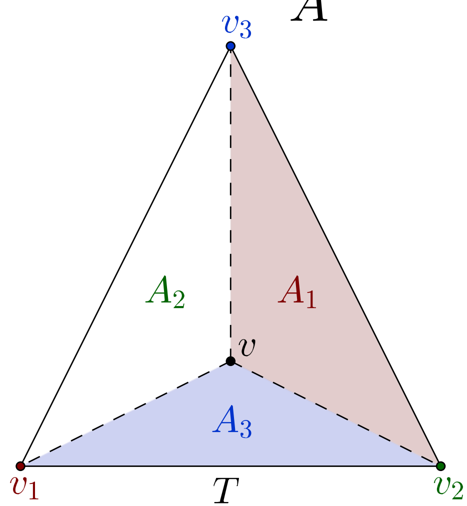
- The convex hull of a set is unique, up to collinearities.
- The boundary of the convex hull of a point set is a polygon on a subset of the points.



# BARYCENTRIC COORDINATES FOR SIMPLICES

# Triangle barycentric coordinates

$$b_3 = \frac{A_3}{A}$$



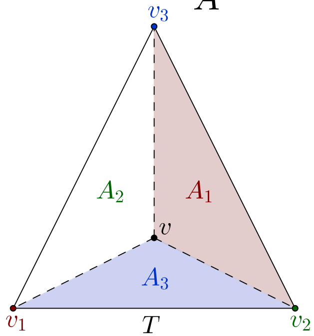

$$b_1 = \frac{A_1}{A} \qquad b_2 = \frac{A_2}{A}$$

$$A = A_1 + A_2 + A_3$$


A. F. Möbius
[1790–1868]

# Triangle barycentric coordinates

$$b_3 = \frac{A_3}{A}$$




$$b_1 = \frac{A_1}{A} \qquad b_2 = \frac{A_2}{A}$$

$$A = A_1 + A_2 + A_3$$

$$\delta_{ij} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases}$$

## Properties:

Constant precision: $b_1 + b_2 + b_3 = 1$

Linear precision: $b_1 v_1 + b_2 v_2 + b_3 v_3 = v$

The Lagrange property: $b_i(v_j) = \delta_{ij}$

Non-negativity: $b_i \geq 0$ for $i = 1, 2, 3$

Linearity along edges: $b_i = av + b$ for $v \in \partial T$

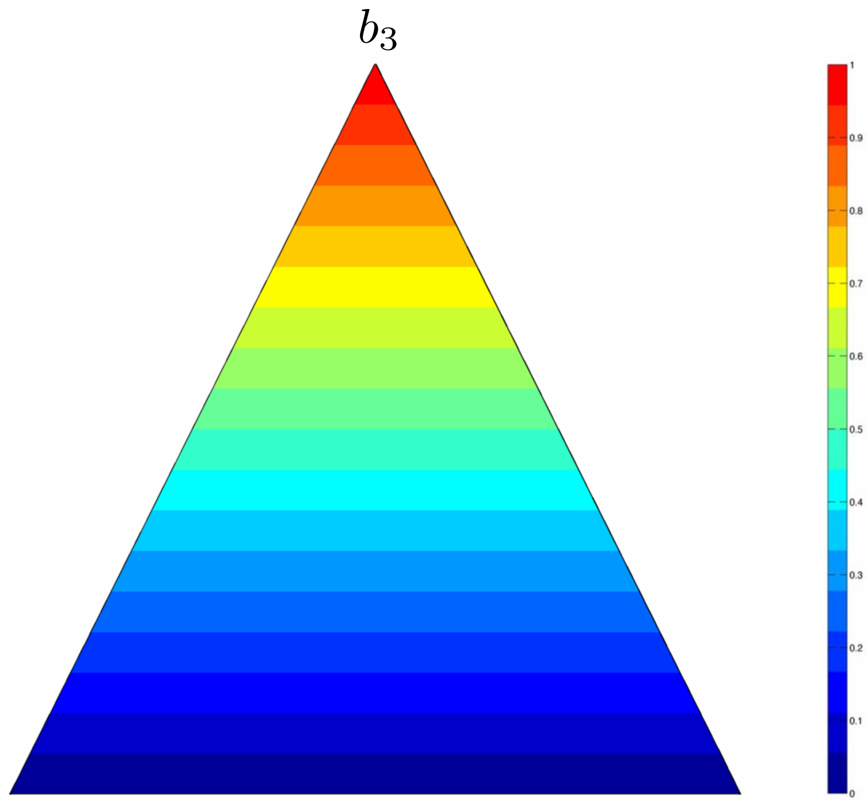
Smoothness: $b_i \in C^k, \ k > 0$

Closed-form: $b_i$ are expressed in analytic form

# Triangle barycentric coordinates

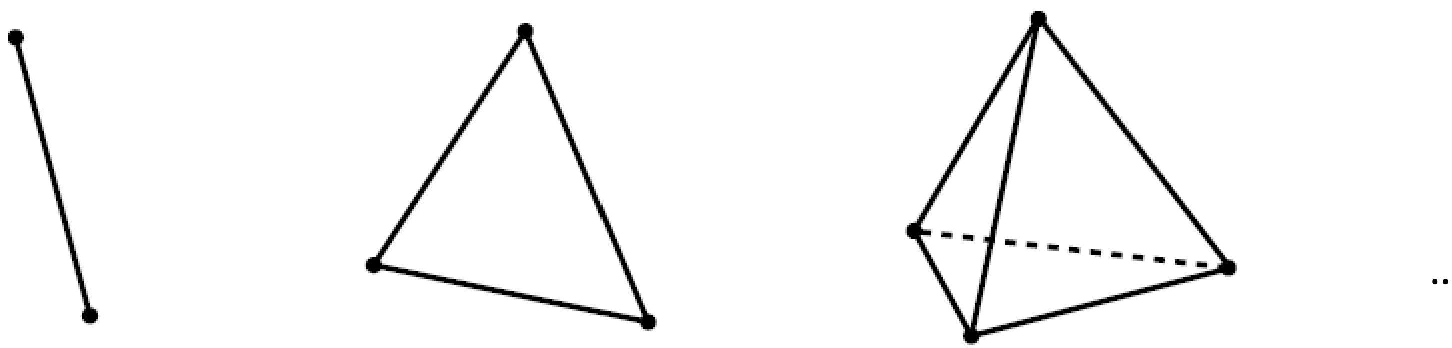
$$\sum_{i=1}^{d+1} b_i(\boldsymbol{p})\boldsymbol{v}_i$$

Data interpolation



Linear function

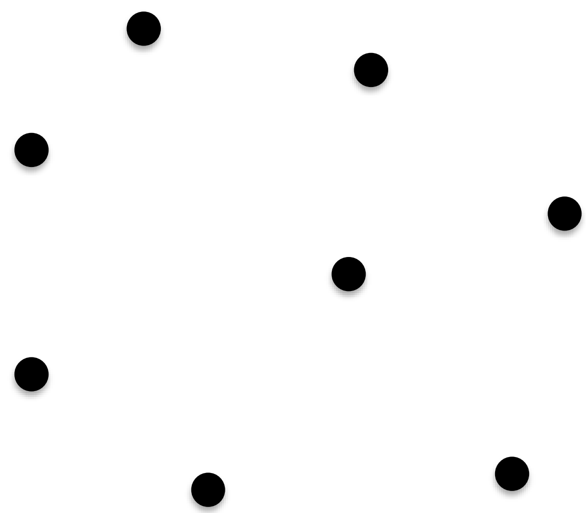
# Barycentric Coordinates for Simplices



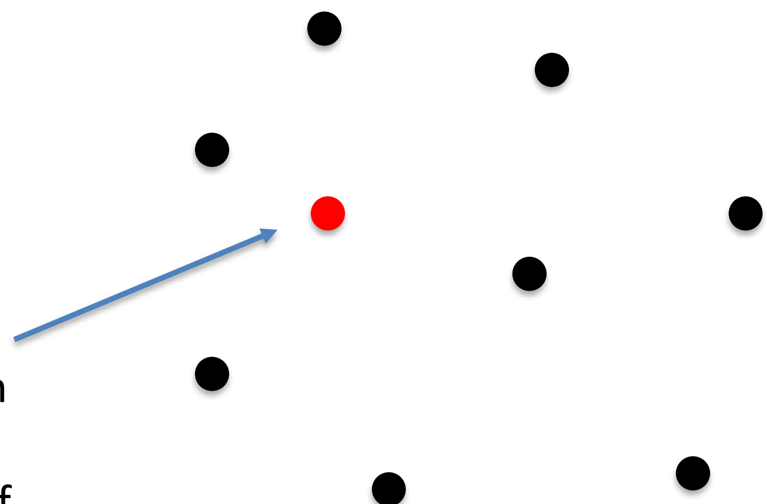
$$\sum_{i=1}^{d+1} b_i(\boldsymbol{p})\boldsymbol{v}_i$$

# BARYCENTRIC COORDINATES FOR POINT SETS

# Point Set



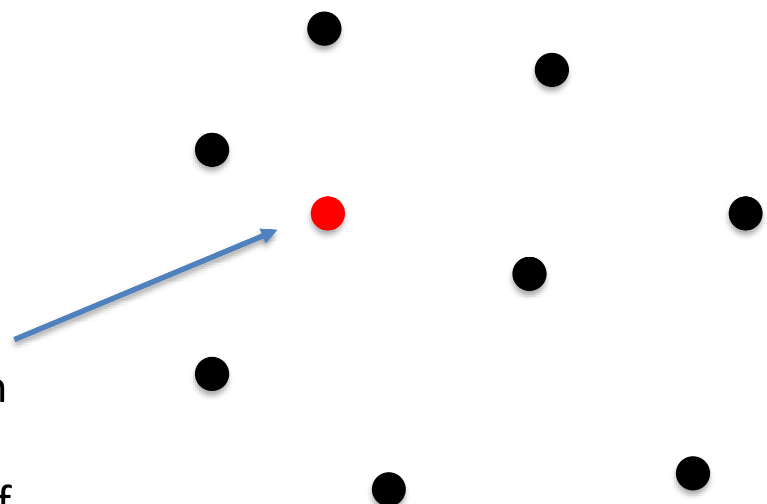
# Barycentric Coordinates?



Weights such that it is a barycenter of the point set?

Lagrange property: $b_i(v_j) = \delta_{ij}$

# Barycentric Coordinates?



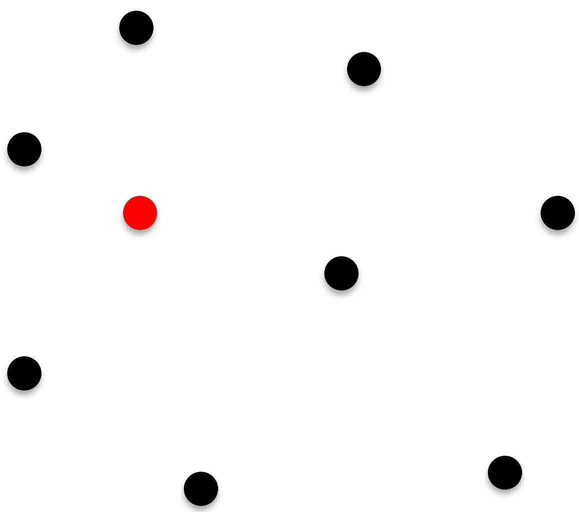
Weights such that it is a barycenter of the point set?

A. F. Möbius
[1790−1868]

Weights always exist if #points >= dimension

Lagrange property: $b_i(v_j) = \delta_{ij}$

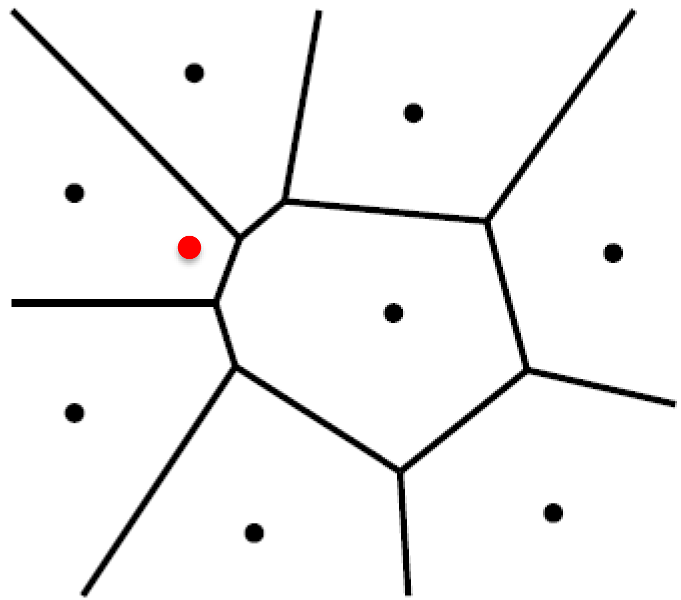
# Inverse Distance [Shepard]



$$u(\mathbf{x}) = \begin{cases} \dfrac{\sum_{i=1}^{N} w_i(\mathbf{x}) u_i}{\sum_{i=1}^{N} w_i(\mathbf{x})}, & \text{if } d(\mathbf{x}, \mathbf{x}_i) \neq 0 \text{ for all } i \\ u_i, & \text{if } d(\mathbf{x}, \mathbf{x}_i) = 0 \text{ for some } i \end{cases}$$

$$w_i(\mathbf{x}) = \frac{1}{d(\mathbf{x}, \mathbf{x}_i)^p}$$

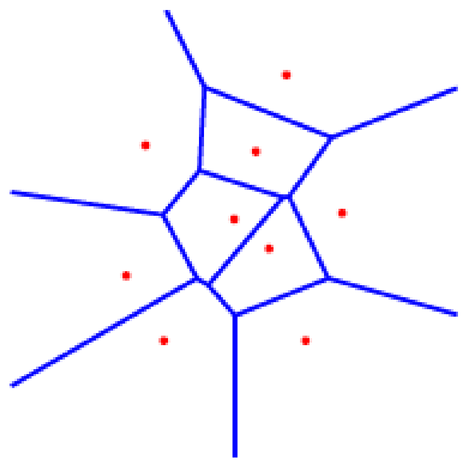
# Natural Neighbor Coordinates [Sibson]



# Voronoi Diagram

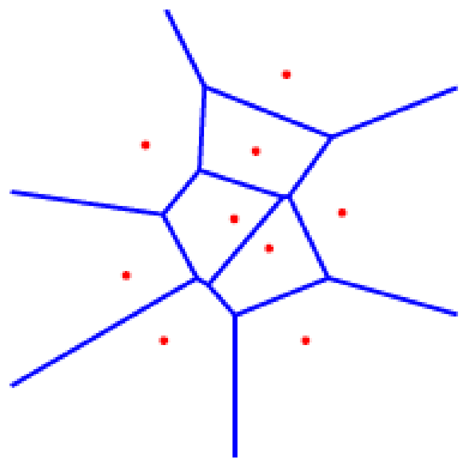
Let $\mathcal{E} = \{\mathbf{p_1}, \ldots, \mathbf{p_n}\}$ be a set of points (so-called sites) in $\mathbb{R}^d$. We associate to each site $\mathbf{p_i}$ its Voronoi region $V(\mathbf{p_i})$ such that:

$$V(\mathbf{p_i}) = \{\mathbf{x} \in \mathbb{R}^d : \|\mathbf{x} - \mathbf{p_i}\| \leq \|\mathbf{x} - \mathbf{p_j}\|, \forall j \leq n\}.$$



# Voronoi Diagram

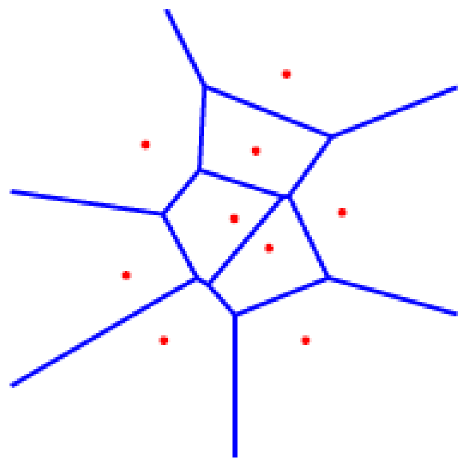
- The collection of the non-empty Voronoi regions and their faces, together with their incidence relations, constitute a cell complex called the **Voronoi diagram** of E.
- The locus of points which are equidistant to two sites pi and pj is called a **bisector**, all bisectors being affine subspaces of $IR^d$ (lines in 2D).



demo

# Voronoi Diagram

- A Voronoi cell of a site *pi* defined as the intersection of closed half-spaces bounded by bisectors. Implies: All Voronoi cells are **convex**.



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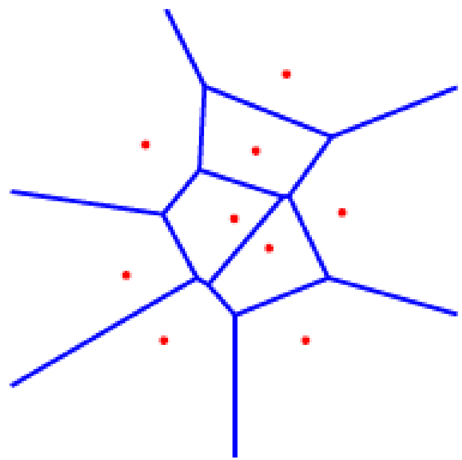
# Voronoi Diagram

- Voronoi cells may be **unbounded** with unbounded bisectors. Happens when a site pi is on the boundary of the convex hull of E.



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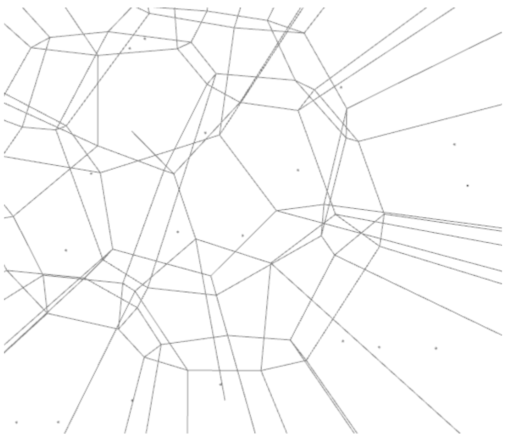
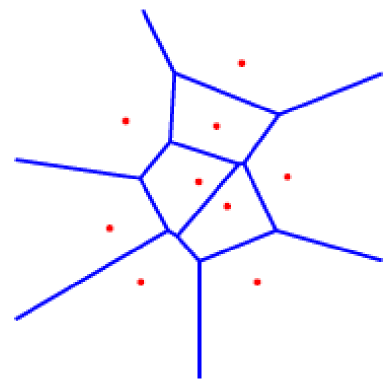
# Voronoi Diagram

- **Voronoi cells** have **faces** of different dimensions.
- In 2D, a face of dimension k is the intersection of 3 - k Voronoi cells. A **Voronoi vertex** is generically equidistant from three points, and a **Voronoi edge** is equidistant from two points.





# Delaunay Triangulation

- Dual structure of the Voronoi diagram.
- The Delaunay triangulation of a set of sites E is a simplicial complex such that k+1 points in E form a Delaunay simplex if their Voronoi cells have nonempty intersection

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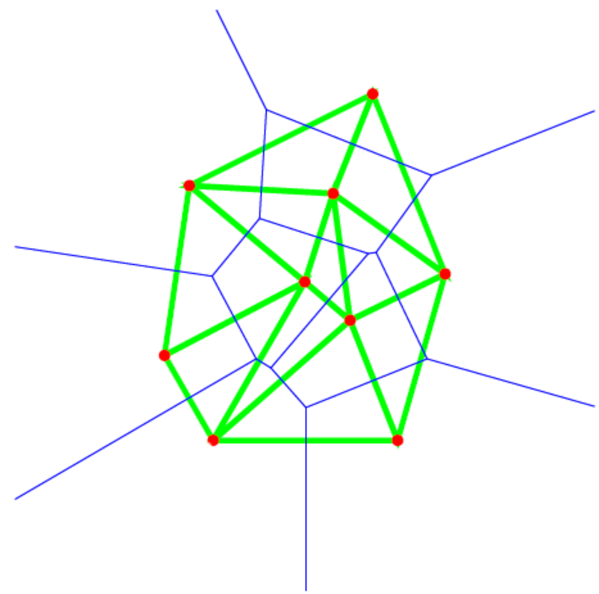
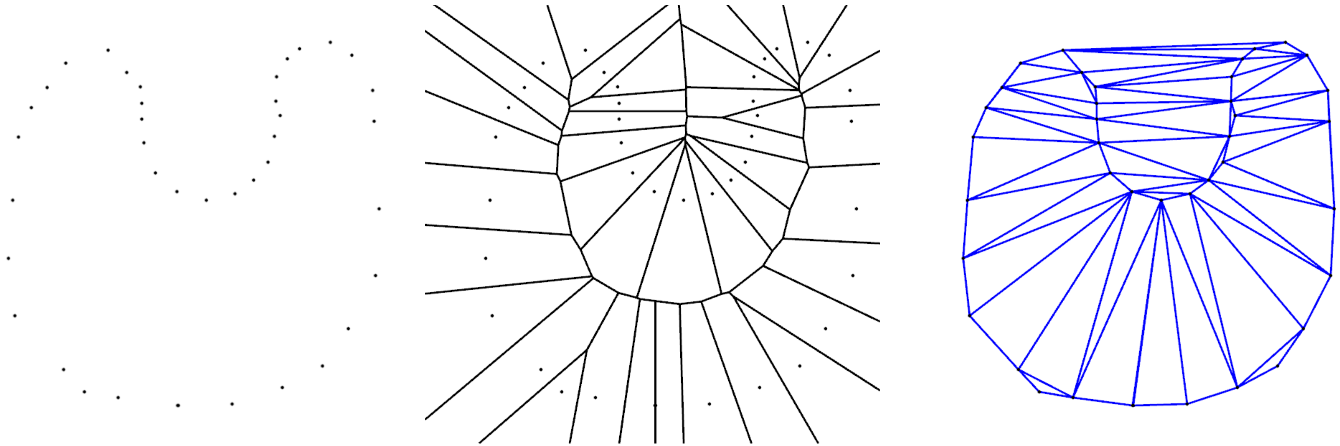


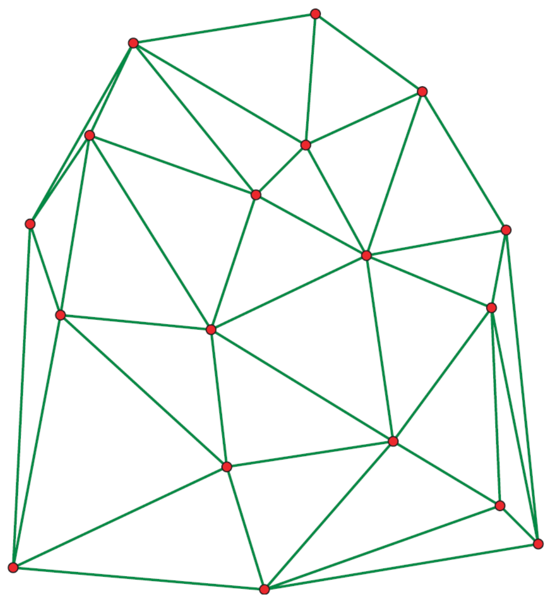
# Delaunay Triangulation

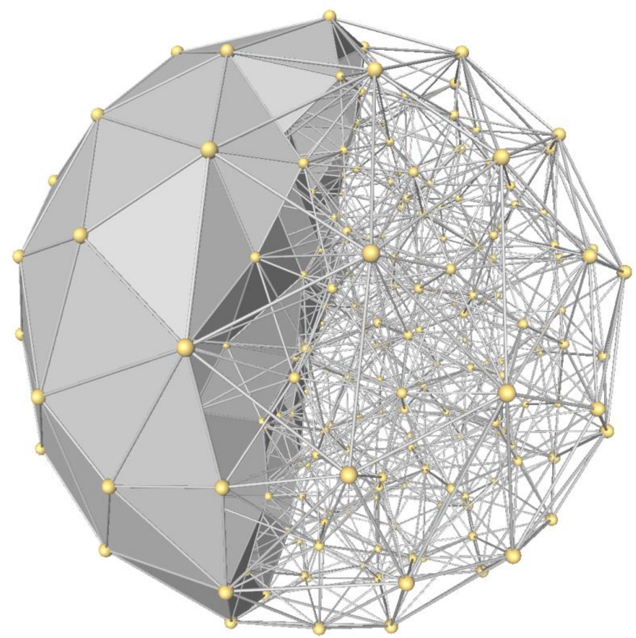
- The Delaunay triangulation of a point set E covers the convex hull of E.



# Delaunay Triangulation

- canonical triangulation associated to any point set





# Delaunay Triangulation: Local Property

- **Empty circle**: A triangulation T of a point set E such that any d-simplex of T has a circumsphere that does not enclose any point of E is a Delaunay triangulation of E. Conversely, any k-simplex with vertices in E that can be circumscribed by a hypersphere that does not enclose any point of E is a face of the Delaunay triangulation of E.

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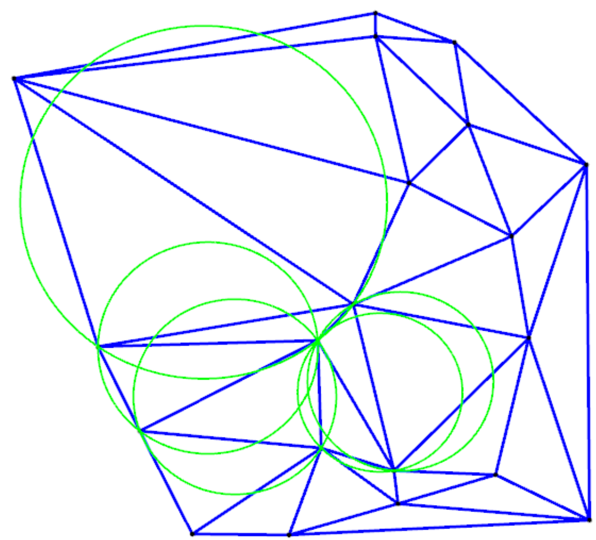
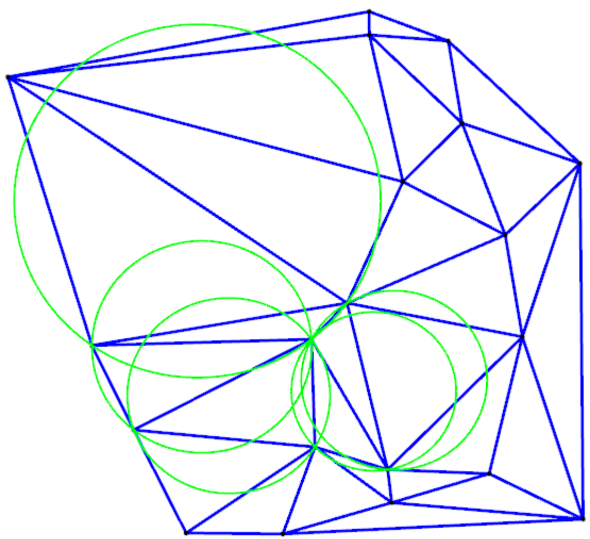


# An naïve O($n^4$) Construction Algorithm

- **Repeat until impossible:**
  - Select a triple of sites.
  - If the circle through them is empty of other sites, keep the triangle whose vertices are the triple.

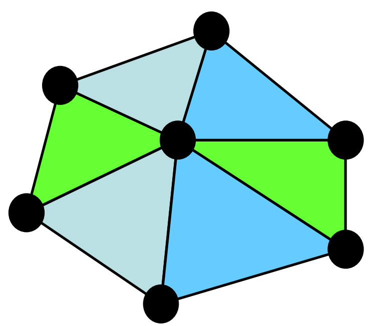
# Naive Algorithm?

- Input: point set
- Output: Delaunay triangles

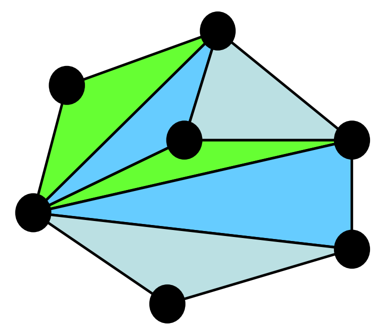


# Delaunay Triangulation

- **In 2D: « quality » triangulation**
  - Smallest triangle angle: The Delaunay triangulation of a point set E is the triangulation of E which **maximizes the smallest angle**.
  - Even stronger: The triangulation of E whose **angular vector** is **maximal** for the lexicographic order is the Delaunay triangulation of E.



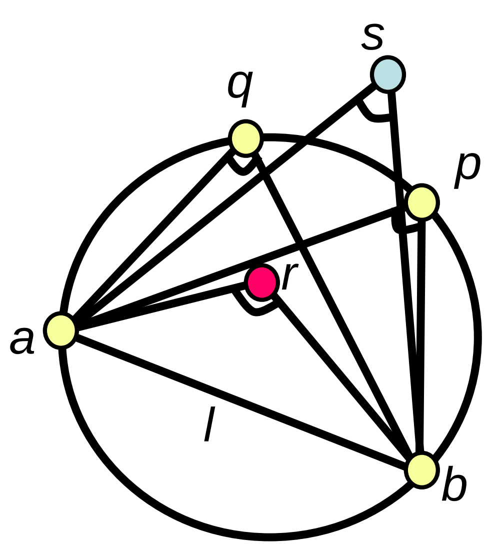
good



bad

# Delaunay Triangulation

- **Thales' Theorem**: Let $C$ be a circle, and $l$ a line intersecting C at points $a$ and $b$. Let $p$, $q$, $r$ and $s$ be points lying on the same side of $l$, where $p$ and $q$ are on $C$, $r$ inside $C$ and $s$ outside $C$. Then:



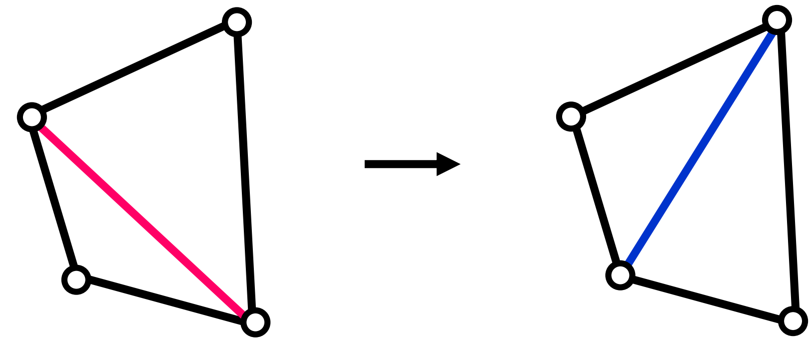

$$\angle arb = 2\angle apb$$

$$\angle aqb > \angle asb$$

# Edge Flipping Algorithm

## Improving a triangulation:

- In any convex quadrangle, an *edge flip* is possible. If this flip *improves* the triangulation locally, it also improves the global triangulation.

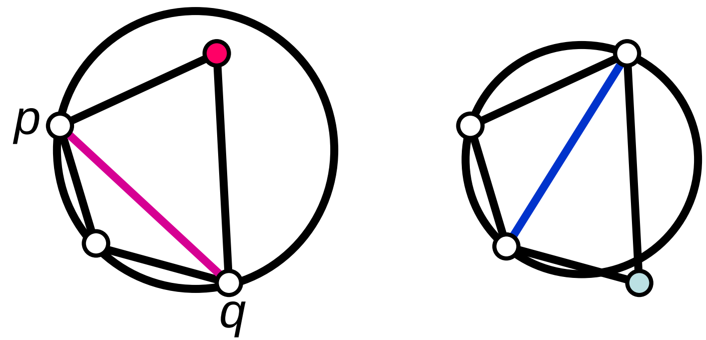


- If an edge flip improves the triangulation, the first edge is called **illegal**.

# Delaunay Triangulation

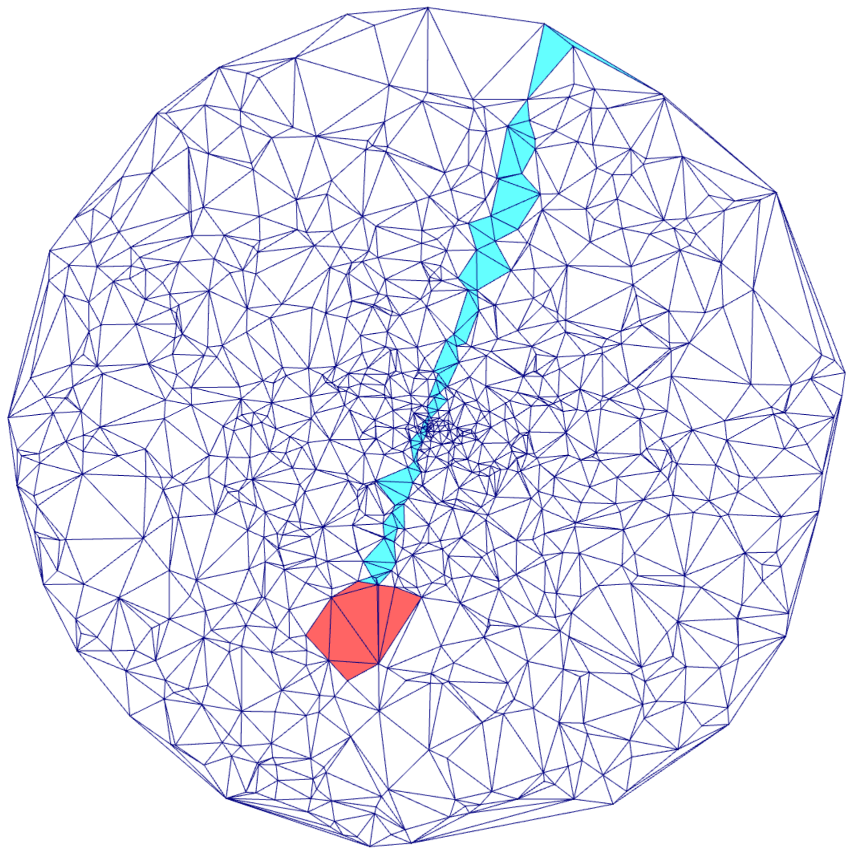
**Illegal edges:**

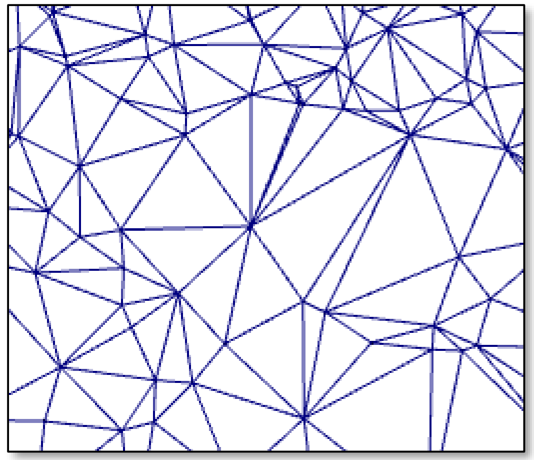
- **Lemma:** An edge *pq* is illegal iff one of its opposite vertices is inside the circle defined by the other three vertices.
- **Proof:** By Thales' theorem.




- **Theorem:** A Delaunay triangulation does not contain illegal edges.
- **Corollary:** A triangle is Delaunay iff the circle through its vertices is empty of other sites (the *empty-circle* condition).
- **Corollary:** The Delaunay triangulation is not unique if more than three sites are co-circular.

# Locate & Star-hole





**Theorem:** If *a*,*b*,*c*,*d* form a CCW convex polygon, then *d* lies in the circle determined by *a*, *b* and *c* iff:

$$\det\begin{pmatrix} a_x & a_y & a_x^2+a_y^2 & 1 \\ b_x & b_y & b_x^2+b_y^2 & 1 \\ c_x & c_y & c_x^2+c_y^2 & 1 \\ d_x & d_y & d_x^2+d_y^2 & 1 \end{pmatrix} > 0$$

**Proof:** We prove that equality holds if the points are co-circular. There exists a center *q* and radius *r* such that:

$$(a_x - q_x)^2 + (a_y - q_y)^2 = r^2$$

and similarly for *b*, *c*, *d*:

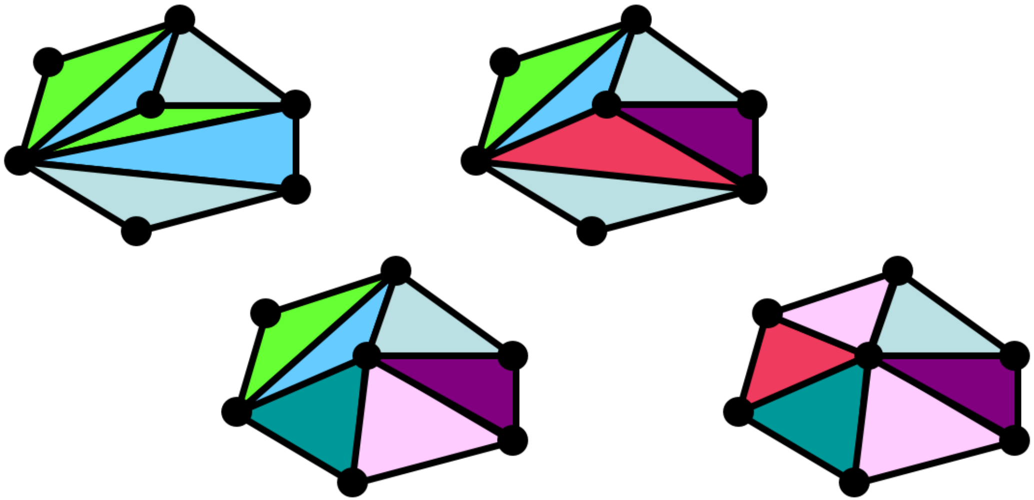
$$\begin{pmatrix} a_x^2+a_y^2 \\ b_x^2+b_y^2 \\ c_x^2+c_y^2 \\ d_x^2+d_y^2 \end{pmatrix} - 2q_x \begin{pmatrix} a_x \\ b_x \\ c_x \\ d_x \end{pmatrix} - 2q_y \begin{pmatrix} a_y \\ b_y \\ c_y \\ d_y \end{pmatrix} + (q_x^2 + q_y^2 - r^2)\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = 0$$

So these four vectors are linearly dependent, hence their det vanishes.

**Corollary:** $d \in$ circle(*a*,*b*,*c*) iff $b \in$ circle(*c*,*d*,*a*) iff $c \notin$ circle(*d*,*a*,*b*) iff $a \notin$ circle(*b*,*c*,*d*)

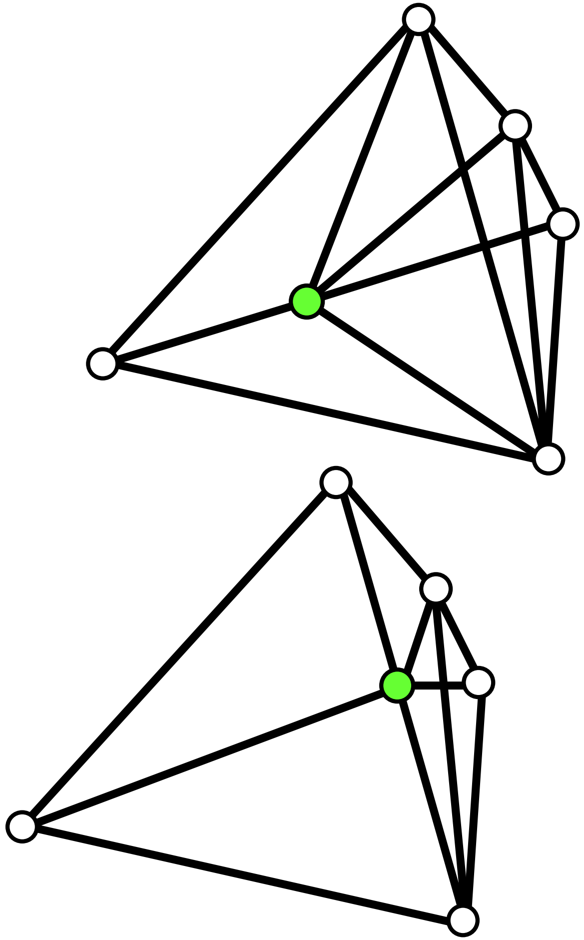
**Another naive construction:**

- Start with an arbitrary triangulation. Flip any illegal edge until no more exist.
- Requires proof that there are no local minima.
- Could take a long time to terminate.

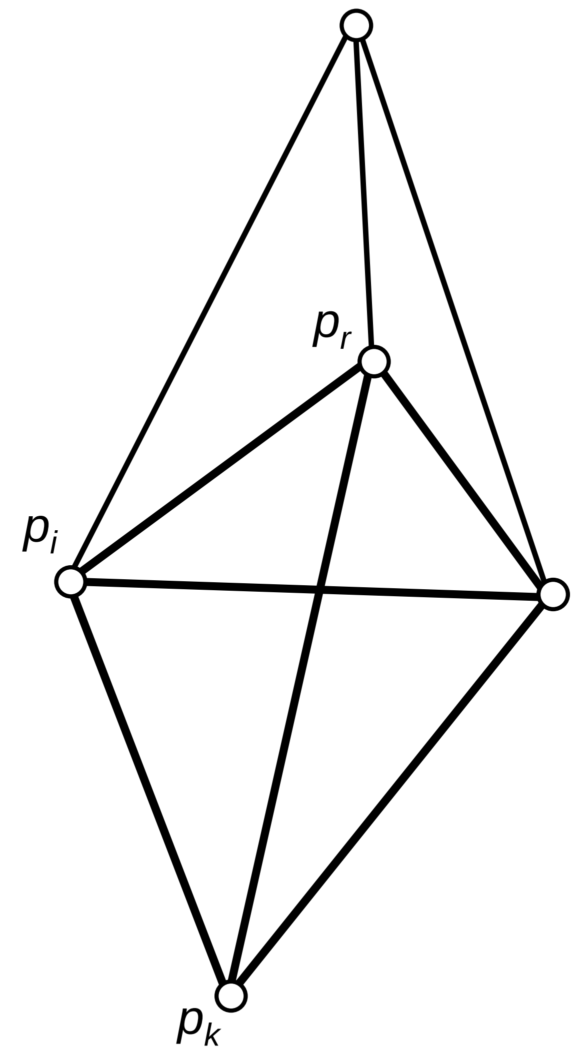


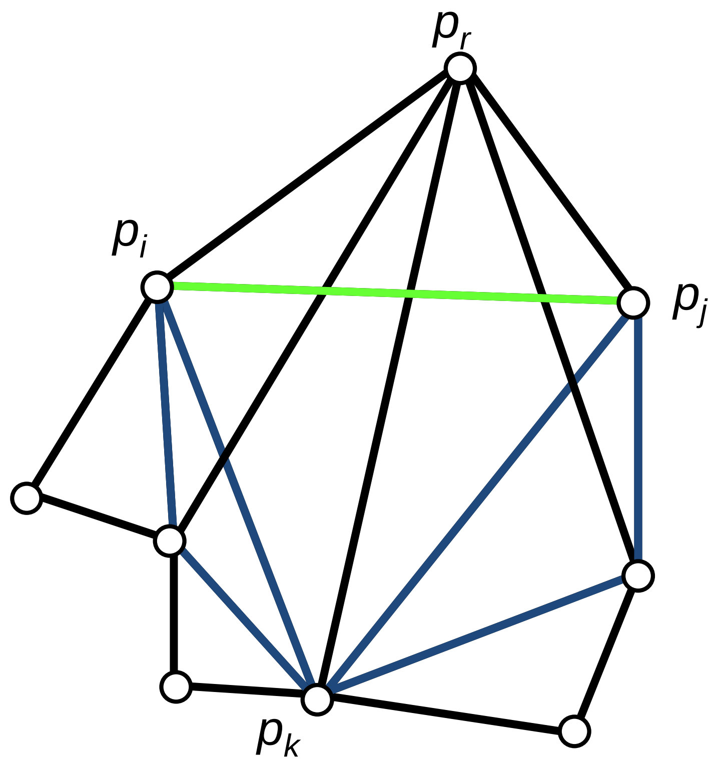
**Incremental algorithm:**

- Form bounding triangle which encloses all the sites.
- Add the sites one after another in random order and update triangulation.
- **If the site is inside an existing triangle:**
  - Connect site to triangle vertices.
  - Check if a 'flip' can be performed on one of the triangle edges. If so – check recursively the neighboring edges.
- **If the site is on an existing edge:**
  - Replace edge with four new edges.
  - Check if a 'flip' can be performed on one of the opposite edges. If so – check recursively the neighboring edges.



- A new vertex $p_r$ is added, causing the creation of edges.
- The legality of the edge $p_i p_j$ (with opposite vertex) $p_k$ is checked.
- If $p_i p_j$ is illegal, perform a flip, and recursively check edges $p_i p_k$ and $p_j p_k$, the new edges opposite $p_r$.
- Notice that the recursive call for $p_i p_k$ cannot eliminate the edge $p_r\, p_k$.
- **Note:** All edge flips replace edges opposite the new vertex by edges incident to it!



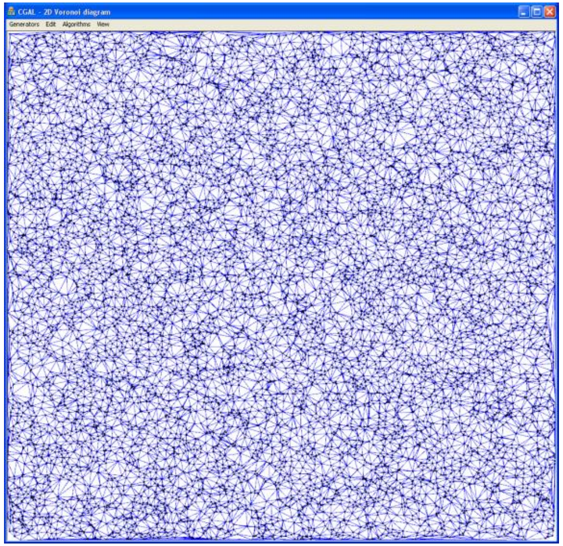


$p_r$
$p_i$
$p_j$
$p_k$

- **Theorem:** The expected number of edges flips made in the course of the algorithm (some of which also disappear later) is at most $6n$.
- **Proof:** During insertion of vertex $p_i$, $k_i$ new edges are created: 3 new initial edges, and $k_i$-3 due to flips.

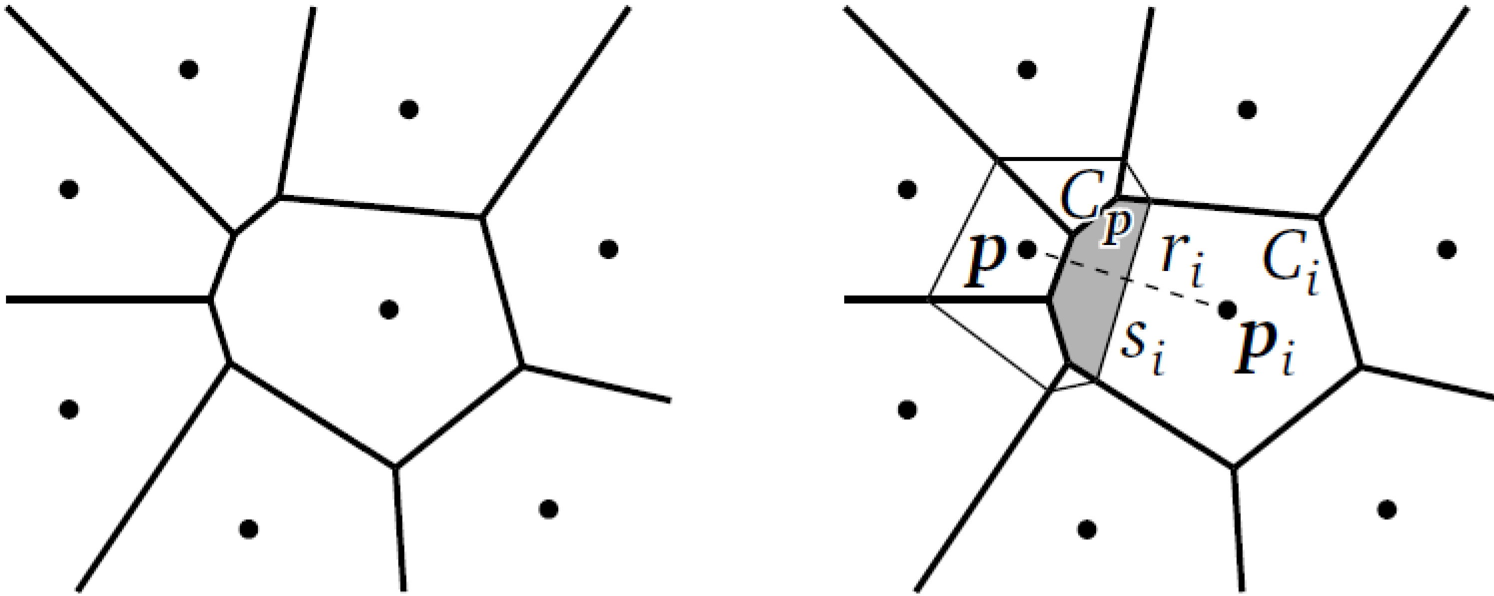
  **Backward analysis**: $E[k_i]$ = the expected degree of $p_i$ after the insertion is complete = 6 (Euler).

- **Point location for every point:** O(log $n$) time.
- **Flips:** Θ(n) expected time in total (for all steps).
- **Total expected time:** O($n$ log $n$).
- **Space:** Θ($n$).



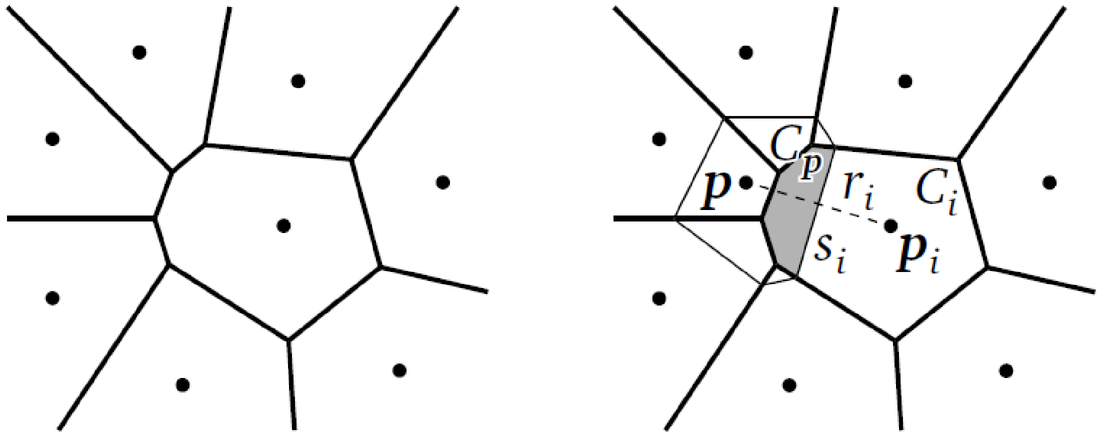

demo

# Sibson Coordinates




$$w_i = \text{Area}[C_i \cap C_p], \qquad i = 1, \ldots, n.$$

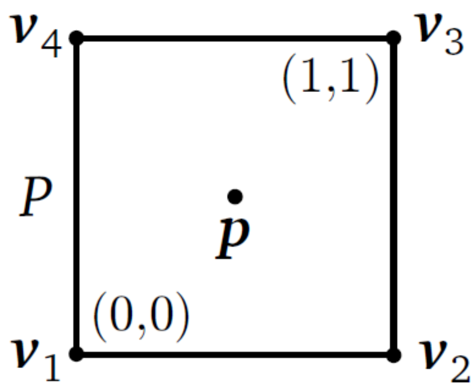
# Sibson Coordinates




- Well define over convex hull
- Local support
- Satisfy Lagrange property
- $C^1$ continuity, except at points $p_i$ (only $C^0$)

# GENERALIZED BARYCENTRIC COORDINATES

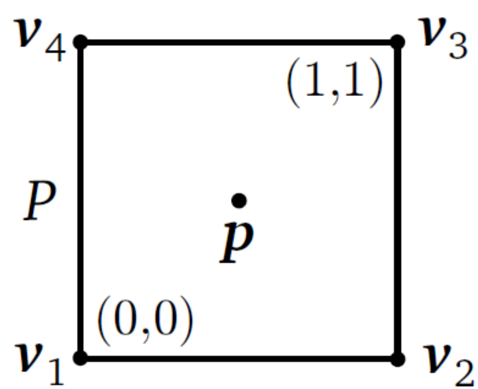
# Quadrilateral?




$$b_1(\boldsymbol{p}) = (1-s)(1-t), \quad b_2(\boldsymbol{p}) = s(1-t), \quad b_3(\boldsymbol{p}) = st, \quad b_4(\boldsymbol{p}) = (1-s)t$$

Bilinear interpolation

# Quadrilateral




$$b_1(\boldsymbol{p}) = (1-s)(1-t), \quad b_2(\boldsymbol{p}) = s(1-t), \quad b_3(\boldsymbol{p}) = st, \quad b_4(\boldsymbol{p}) = (1-s)t$$

Bilinear map on unit square

# Quadrilateral

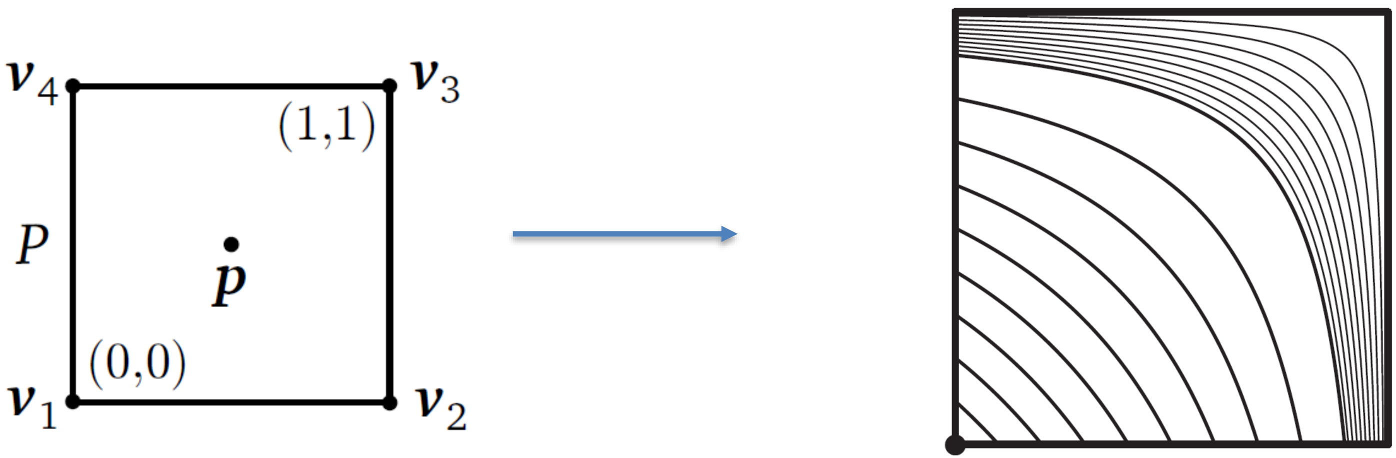



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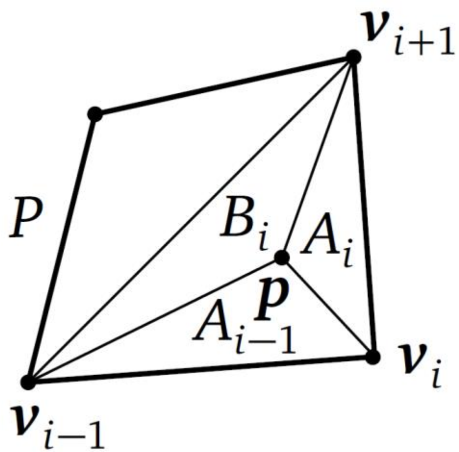
Image of bilinear map on unit square

# Unified Formula! [Floater]

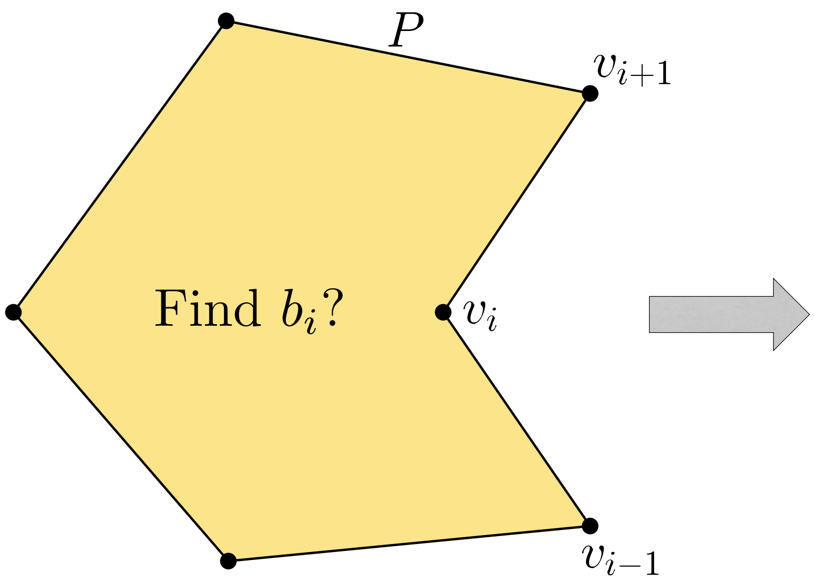



$$b_i(\boldsymbol{p}) = \frac{4A_{i+1}(\boldsymbol{p})A_{i+2}(\boldsymbol{p})}{G_{i+1}(\boldsymbol{p})G_{i+2}(\boldsymbol{p})}, \qquad i = 1, \ldots, 4$$

$$G_i = 2A_i - B_i - B_{i+1} + \sqrt{B_1^2 + B_2^2 + 2A_1A_3 + 2A_2A_4},$$

signed area

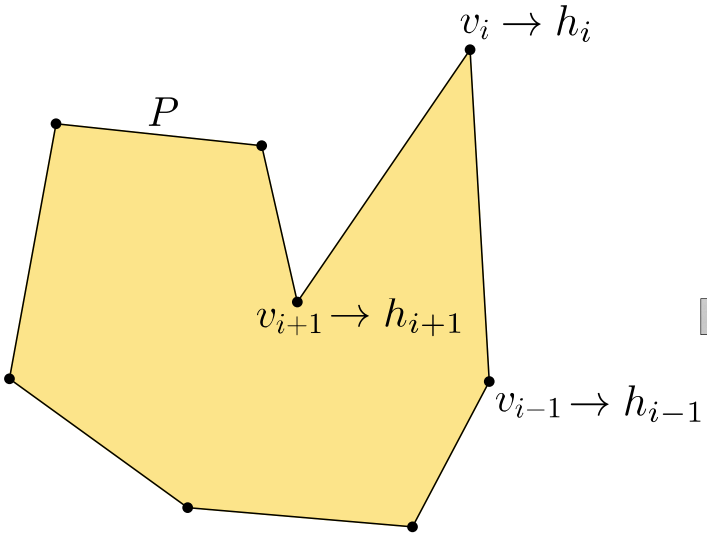
# Generalized barycentric coordinates



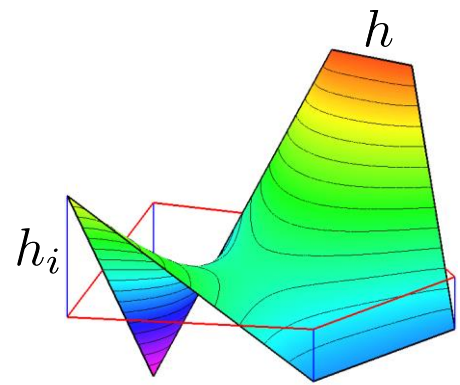

Properties:

- Constant precision
- Linear precision
- The Lagrange property
- Non-negativity
- Linearity along edges
- Smoothness
- Closed-form

# Interpolation




$$\sum_{i=1}^{n} b_i h_i = h$$



## Generalized barycentric coordinates

$$b_i = \frac{w_i}{W}, \text{ where } W = \sum_{j=1}^{n} w_j$$

Different weights -> different coordinate functions

# Generalized barycentric coordinates

- Wachspress coordinates [Wachspress, 1975]
- Discrete harmonic coordinates [Pinkall and Polthier, 1993]
- Mean value coordinates [Floater, 2003]
- Metric coordinates [Malsch et al., 2005]
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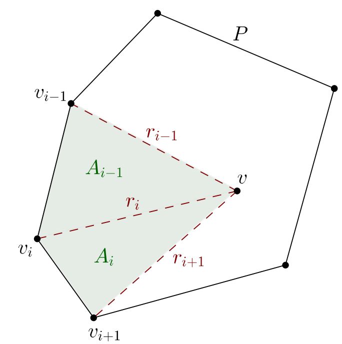
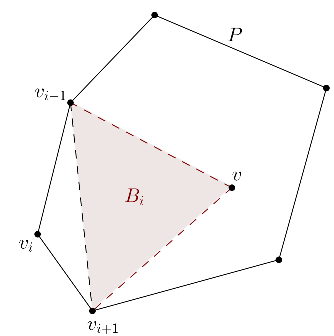
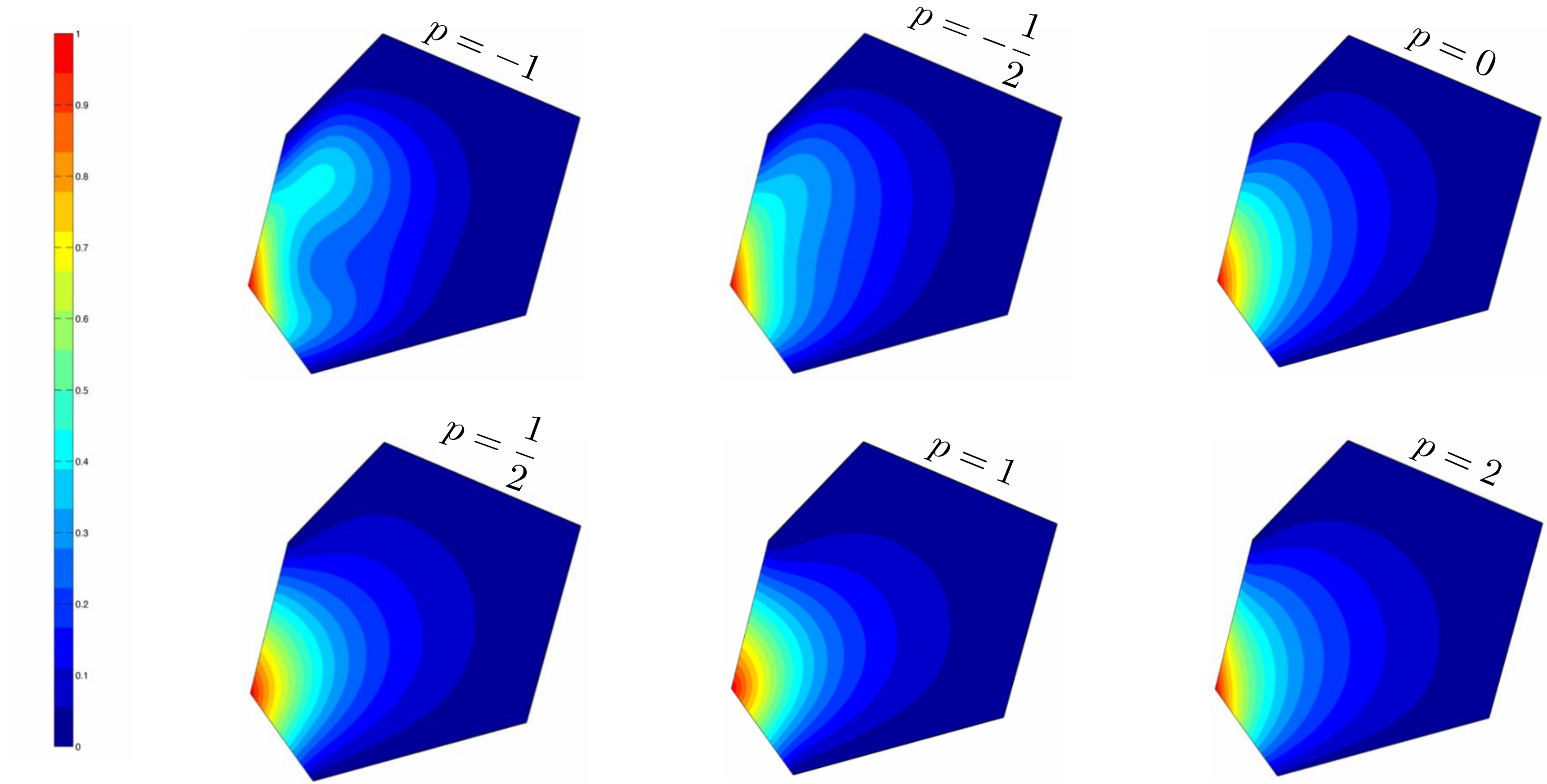



- Three-point coordinates [Floater et al., 2006]

$$p \in \mathbb{R}, \qquad w_i = \frac{r_{i-1}^p A_i - r_i^p B_i + r_{i+1}^p A_{i-1}}{A_{i-1} A_i}$$



# Three-point coordinates



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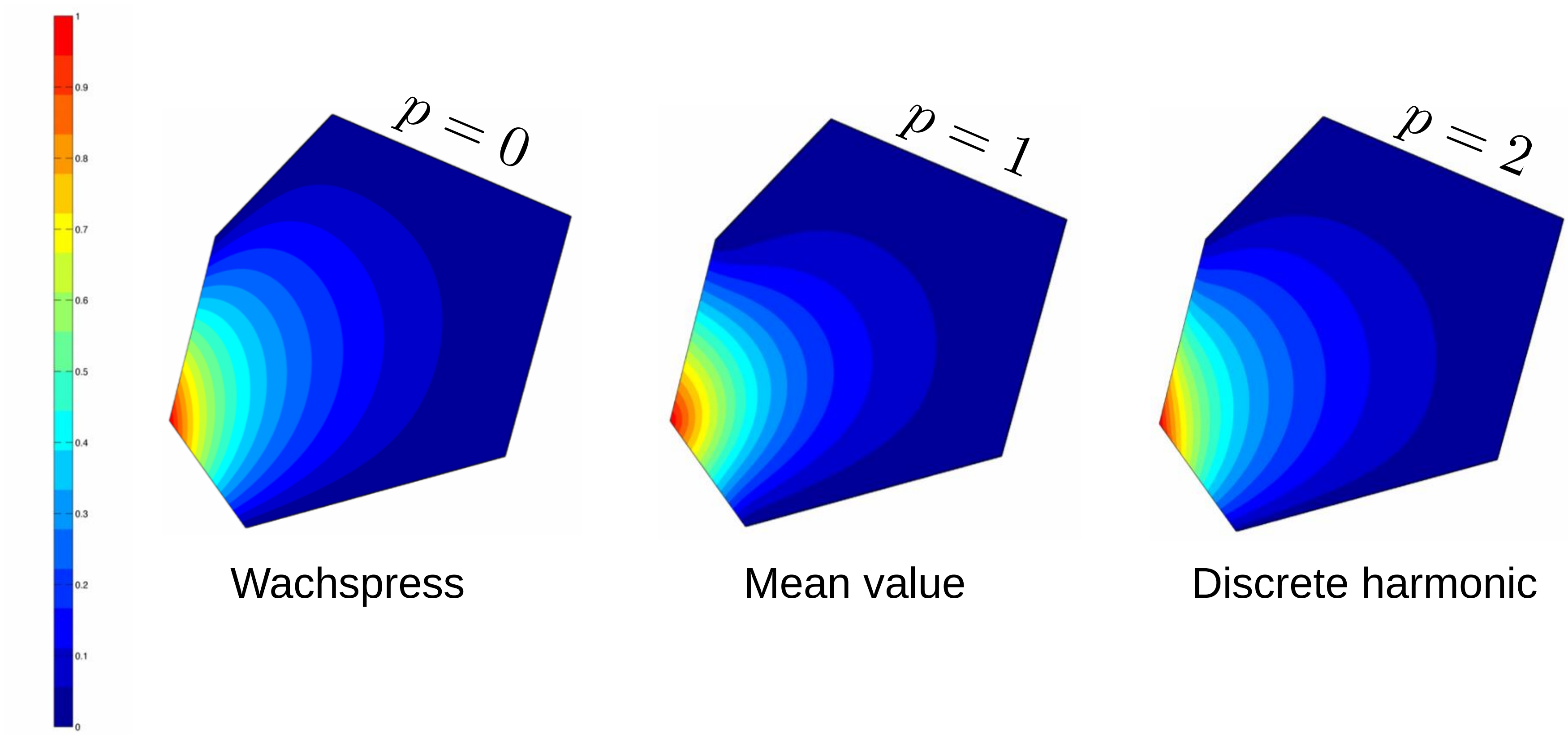
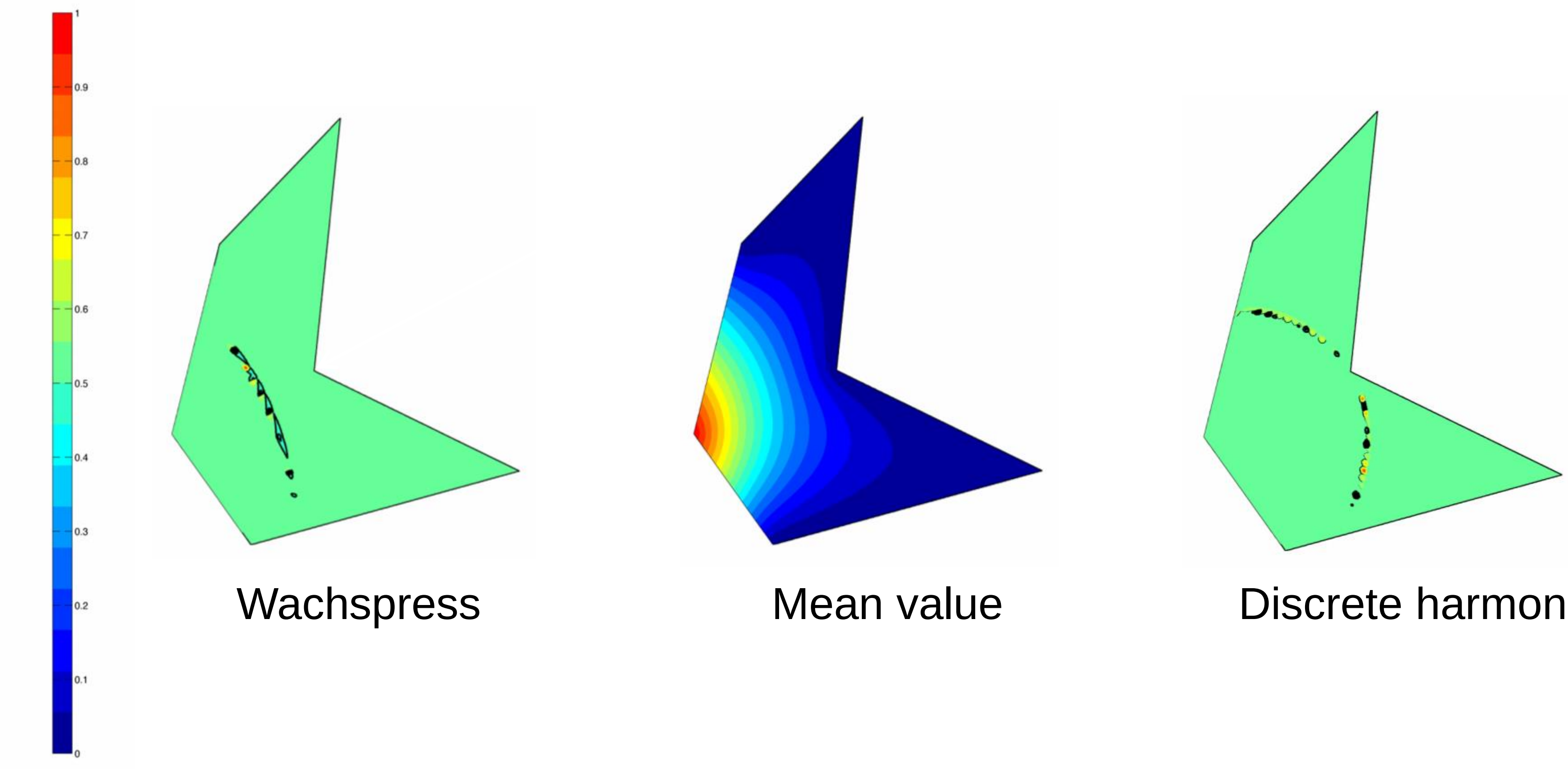


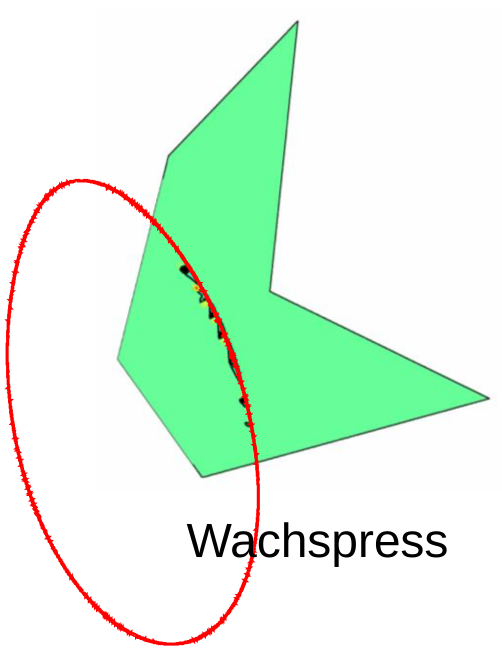

Wachspress Mean value Discrete harmonic

# Three-point coordinates



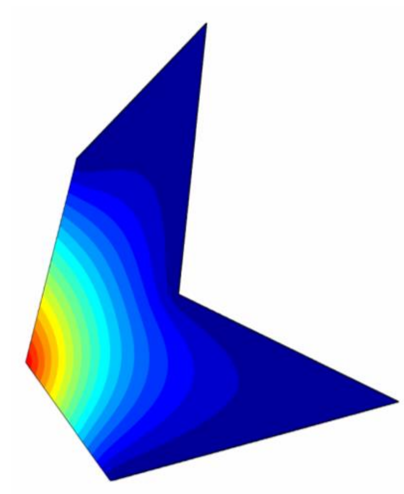

Wachspress    Mean value    Discrete harmonic

# Three-point coordinates



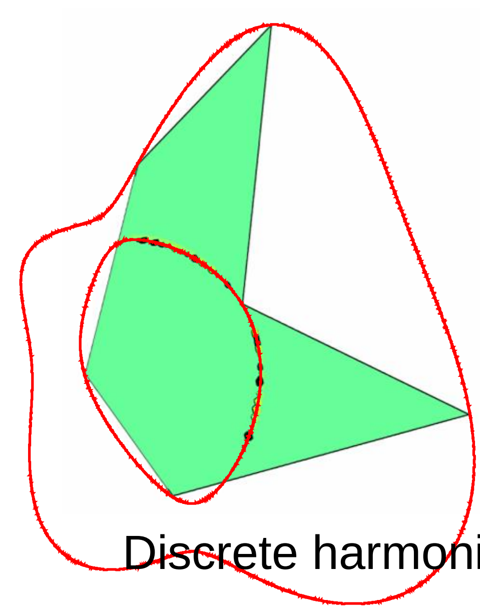
Wachspress

$W = 0$



Mean value

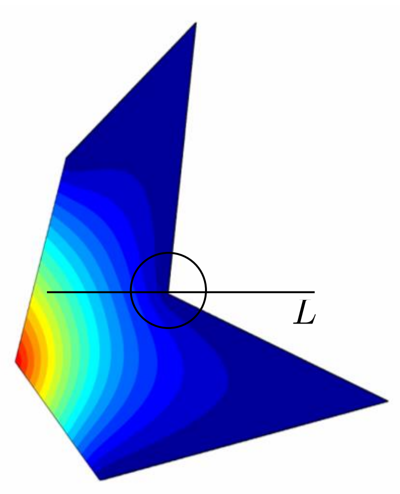
$$b_i = \frac{w_i}{W}$$



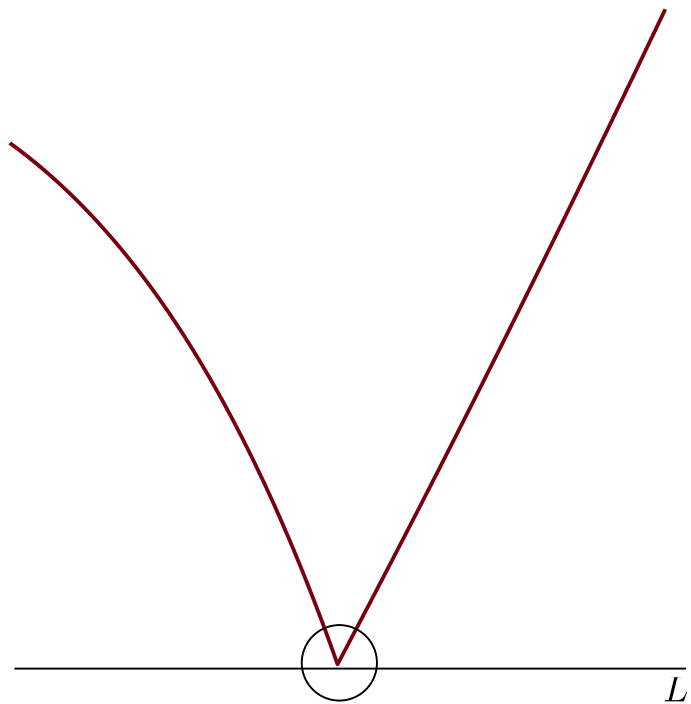
Discrete harmonic

$W = 0$

# Mean value coordinates



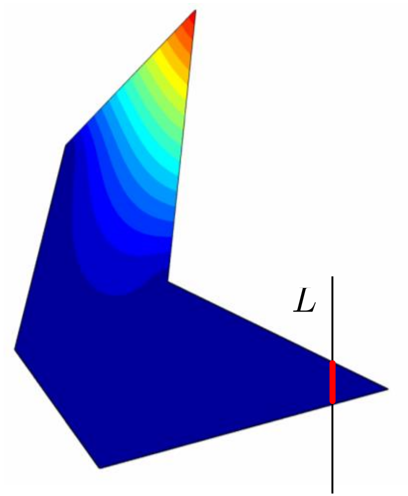

Mean value



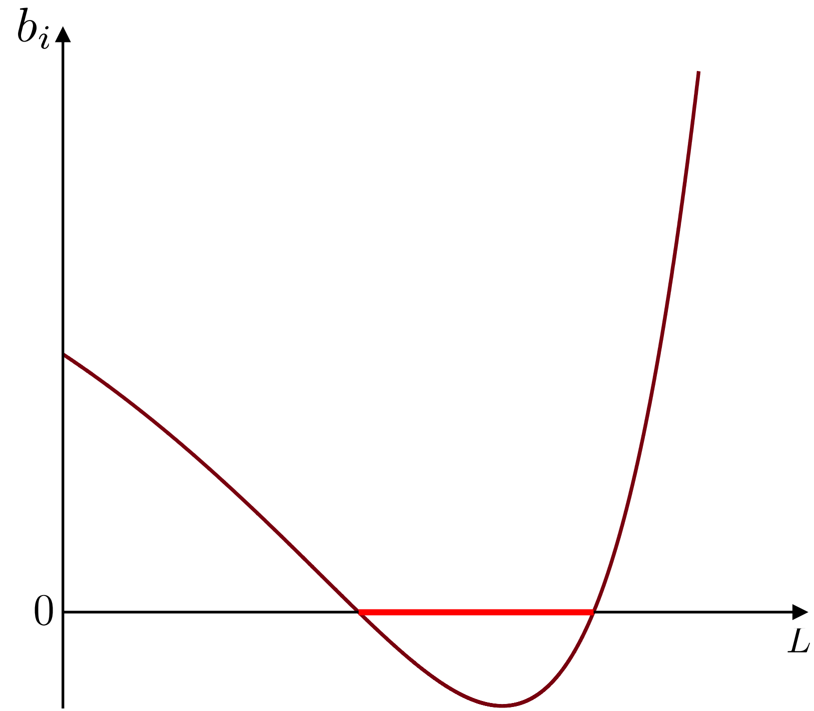

$C^0$ continuity

# Mean value coordinates

## Positive mean value coordinates [Lipman et al., 2007]




Mean value



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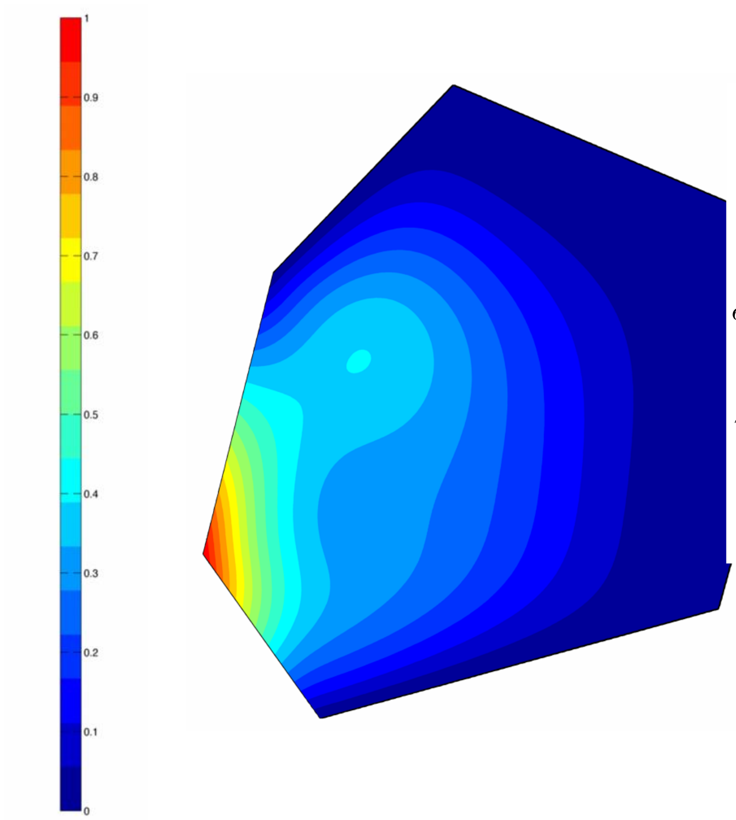
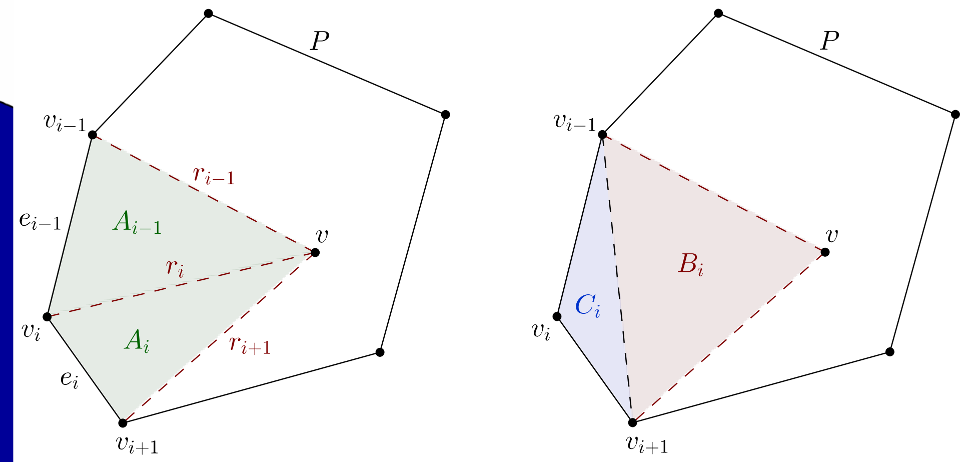
# Metric coordinates

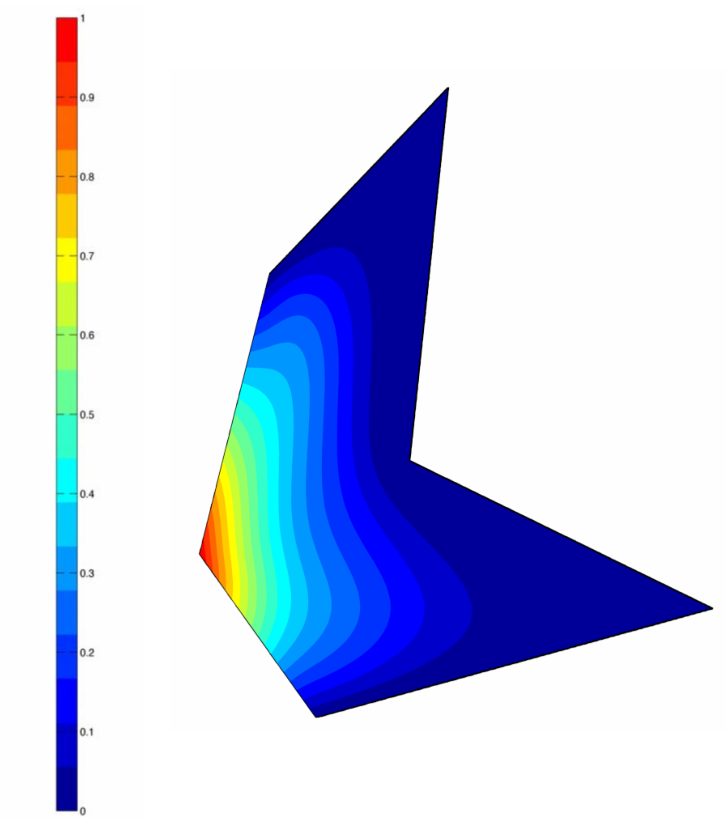


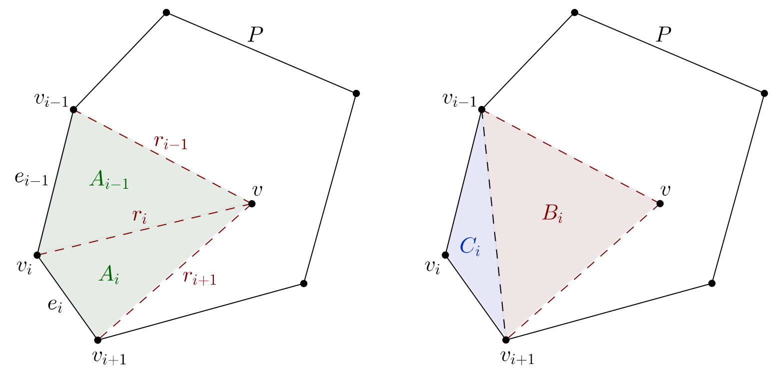




$$q_i = r_i + r_{i+1} - e_i$$

$$w_i = \frac{A_{i-2}}{C_{i-1} q_{i-2} q_{i-1}} - \frac{B_i}{C_i q_{i-1} q_i} + \frac{A_{i+1}}{C_{i+1} q_i q_{i+1}}$$

# Metric coordinates

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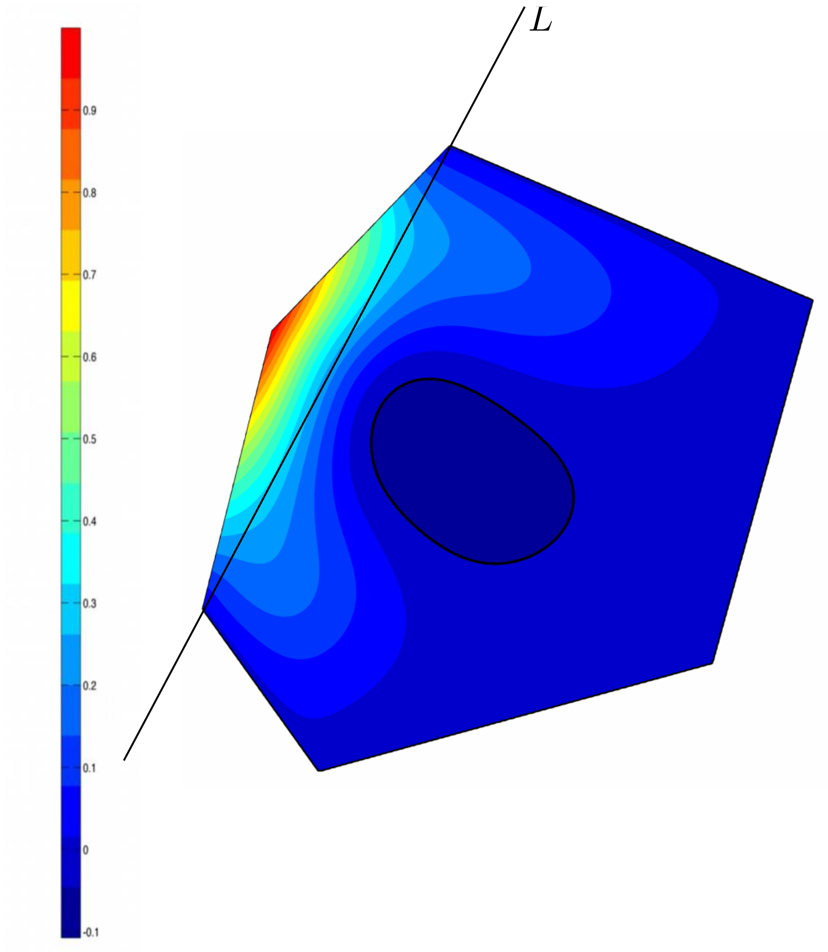


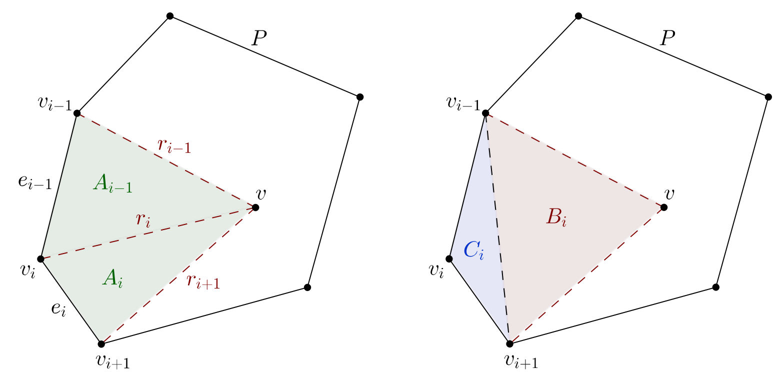


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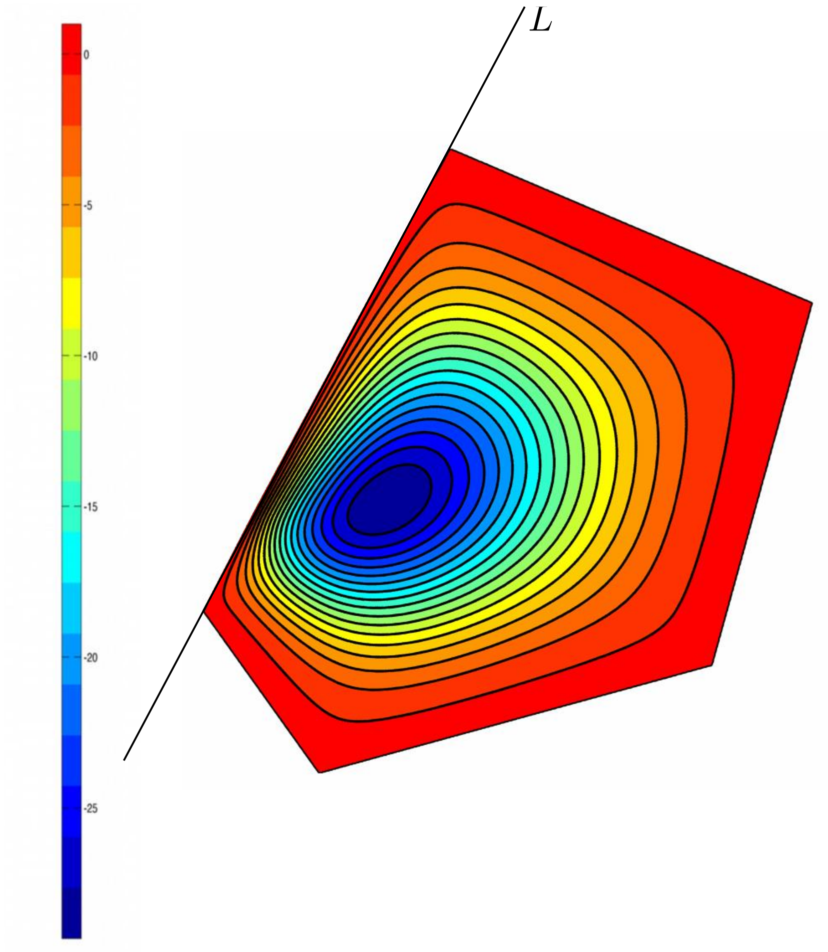


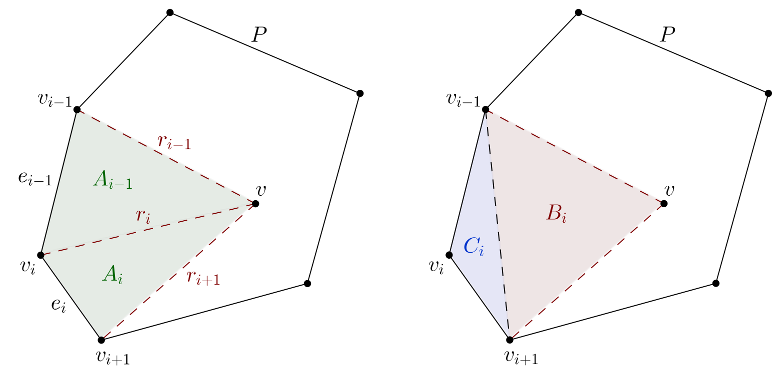


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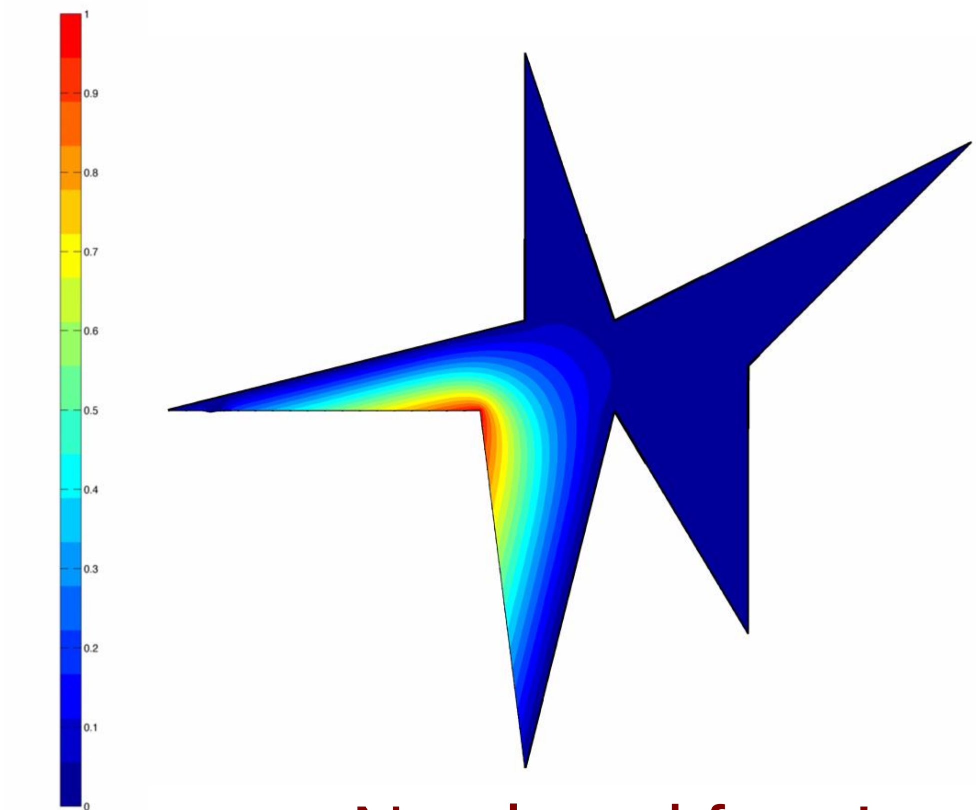
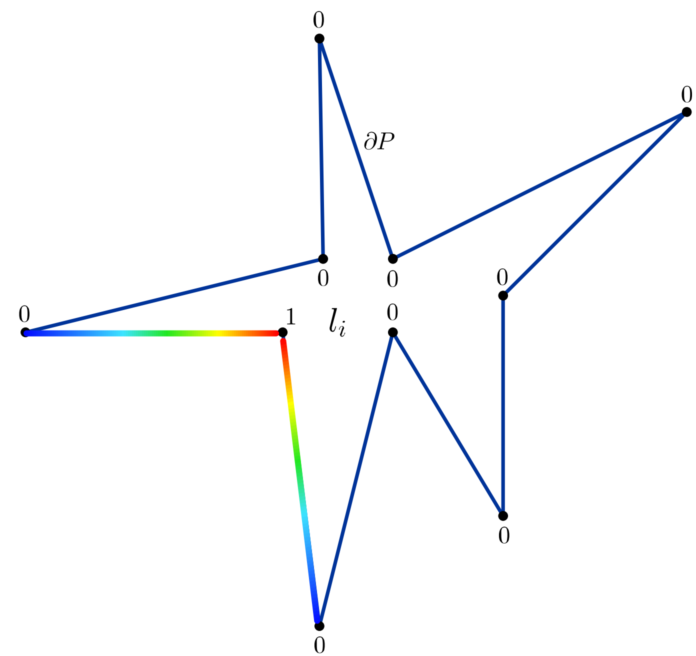
# Harmonic coordinates




No closed form!

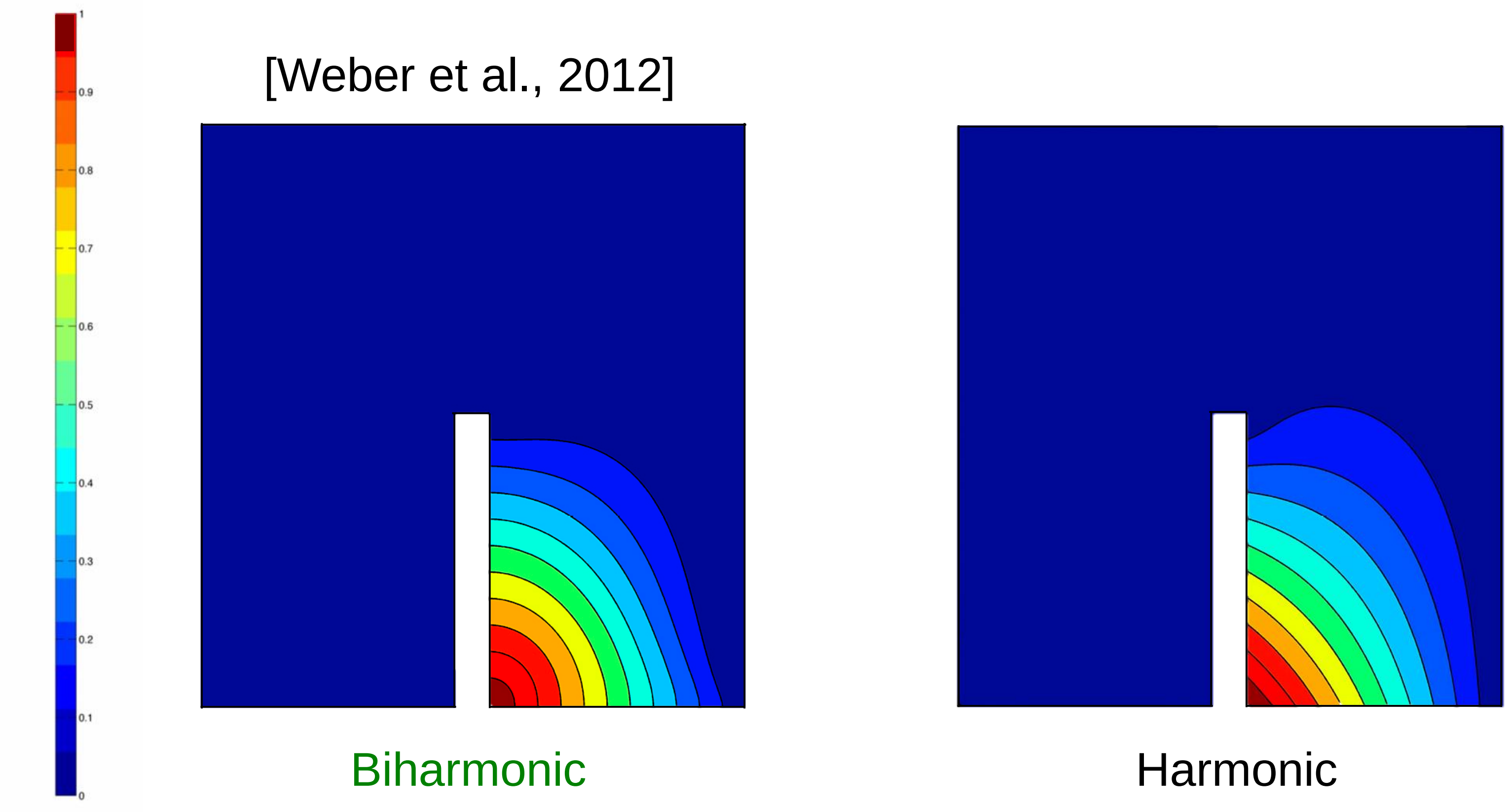
Solve the Laplacian equation:

$$\Delta b_i = 0 \text{ s.t. } b_i|_{\partial P} = l_i$$



# Biharmonic vs Harmonic

[Weber et al., 2012]




Biharmonic

Harmonic

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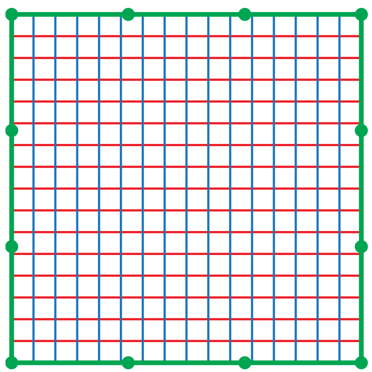
# Barycentric mapping

Source polygon



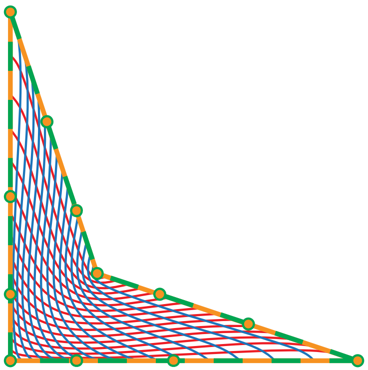
$$P = [v_1, \ldots, v_n]$$

$$v_i \in \mathbb{R}^2$$

For $v \in P$ :

$$f(v) = \sum_{i=1}^{n} b_i(v)\hat{v}_i$$

Target polygon

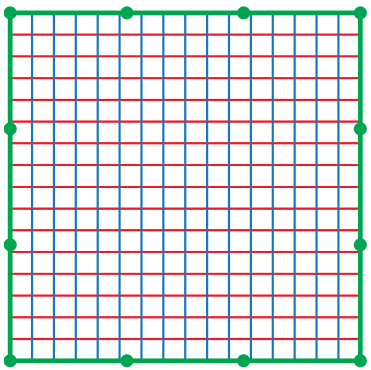


$$\hat{P} = [\hat{v}_1, \ldots, \hat{v}_n]$$

$$\hat{v}_i \in \mathbb{R}^2$$

# Complex barycentric mapping

Source polygon



$P = [z_1, \ldots, z_n]$

$z_i \in \mathbb{C}$

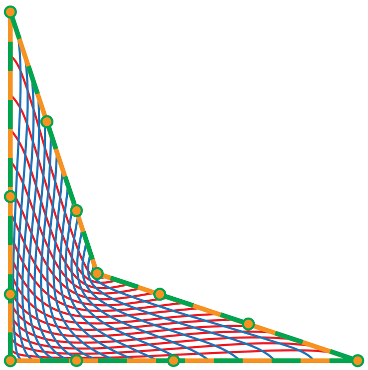
For $z \in P$ :

$$g(z) = \sum_{i=1}^{n} c_i(z)\hat{z}_i$$

with complex barycentric coordinates

$c_i : P \to \mathbb{C}$

Target polygon



$\hat{P} = [\hat{z}_1, \ldots, \hat{z}_n]$

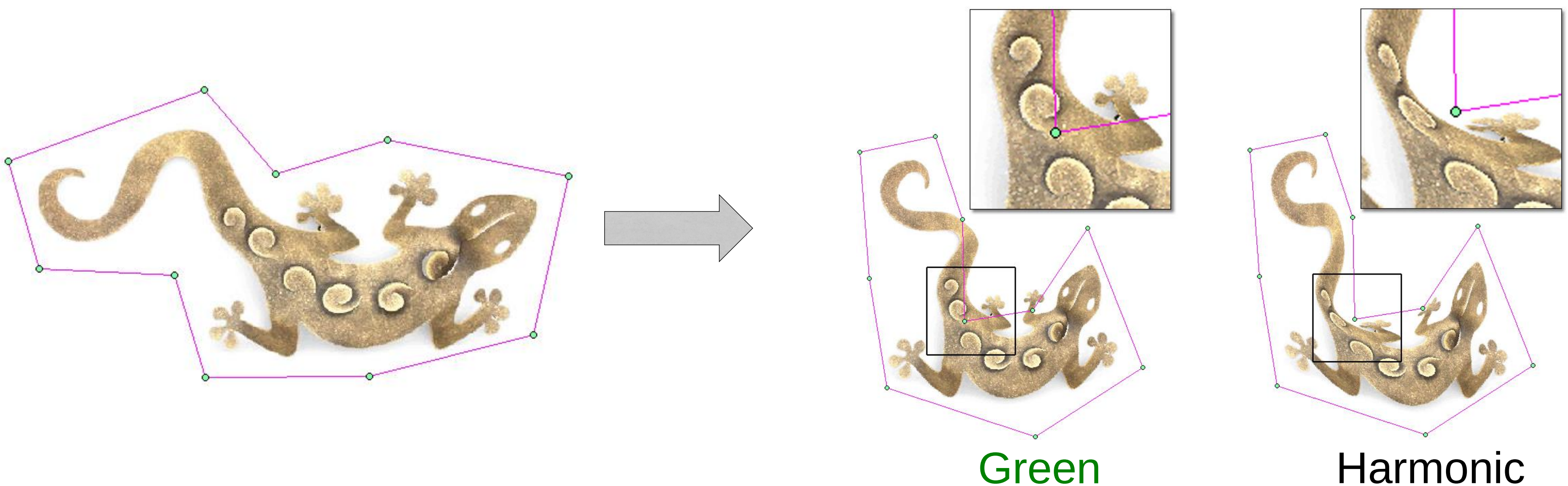
$\hat{z}_i \in \mathbb{C}$

# Complex barycentric coordinates

Three-point coordinates are generalized to complex three-point coordinates

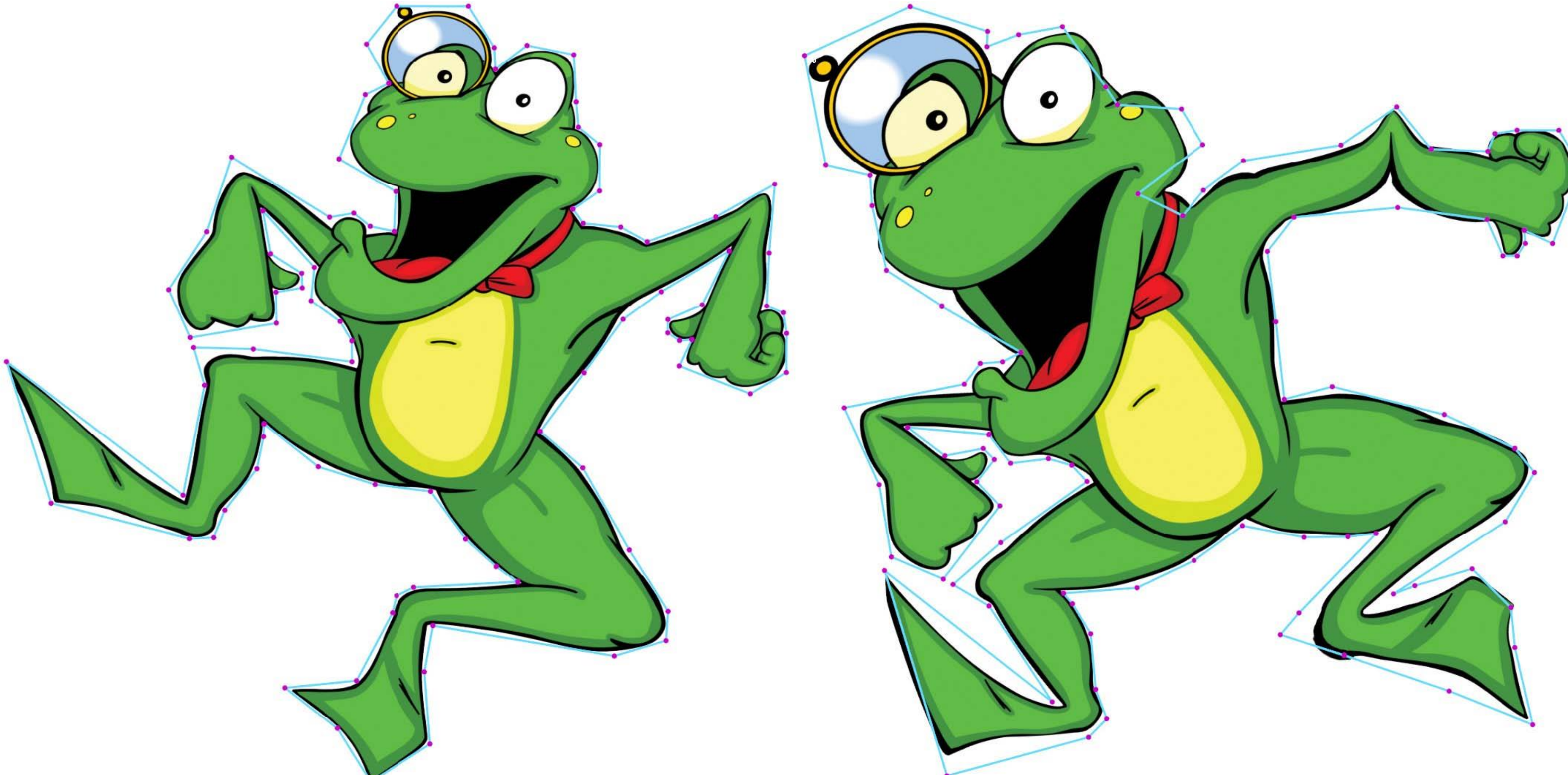
Green coordinates are members of complex three-point coordinates [Lipman et al., 2008]

Induce conformal mappings



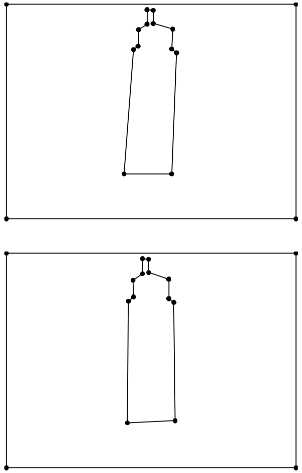
# APPLICATIONS

# Editing



# Image warping


Original image


Mask


Warped image